Shor's Factoring Algorithm

School on Quantum Computing @Yagami

Day 2, Lesson 2

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Outline

- Number theory for factoring
  - Greatest common deviser and Euclidian method
  - Chinese remainder theorem
  - Quadratic equation for factoring
  - Order of a modulo L
- Factoring algorithm
  - Reduction to order finding
  - Continued fractions algorithm
  - Modular exponentiation

The inventor

Peter Shor

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Number theory for factoring

Purpose

To reduce factoring to order finding

1. Greatest common divisor and Euclidian method
2. Chinese remainder theorem
3. Quadratic equation for factoring
4. Order of a modulo L

Greatest common divisor

Definition

The largest integer which is a divisor of two integers a and b is called “greatest common divisor of a and b”, and denoted as

$$\gcd(a, b)$$

If $\gcd(a, b)$ is equal to 1, it is said that “a and b are co-prime”

Example

$$\gcd(9,6) = 3 \qquad \gcd(5,3) = 1$$

Euclidian method

An efficient method for finding the gcd

Example

$$\gcd(494,\ 133) = 19$$

$$494 = 133 \times 3 + 95$$

$$133 = 95 \times 1 + 38$$

$$95 = 38 \times 2 + 19$$

$$38 = 19 \times 2$$

Filling the floor of a rectangular room with square tiles

Chinese remainder theorem

(Below n_1, n_2, s, t, L ... are all positive integers)

Let n_1 and n_2 be co-prime, *i.e.*,

$$\gcd(n_1, n_2) = 1$$

p and q are the remainders of n_1 and n_2, respectively, *i.e.*,

$$0 \le p \le n_1 - 1$$

$$0 \le q \le n_2 - 1$$

Then there *exists* a *unique* s $(1 \le s \le n_1 n_2)$ that satisfies

$$s \equiv p \pmod{n_1}$$

$$s \equiv q \pmod{n_2}$$

Chinese remainder theorem

Proof of uniqueness

Suppose there exists t $(1 \le t \le n_1 n_2, t < s)$ that satisfies

$$t \equiv p \pmod{n_1}$$

$$t \equiv q \pmod{n_2}$$

$\gcd(9,15) \ne 1$

$45 \equiv 0 \pmod{9}$

$45 \equiv 0 \pmod{15}$

$45 \not\equiv 0 \pmod{135}$

Then

$$\begin{array}{l} s - t \equiv 0 \pmod{n_1} \\ s - t \equiv 0 \pmod{n_2} \end{array} \Rightarrow s - t \equiv 0 \pmod{n_1 n_2}$$

$\gcd(n_1, n_2) = 1$

This means $s - t \ge n_1 n_2$, which contradicts the assumption $1 \le t < s \le n_1 n_2$

Chinese remainder theorem

Proof of existence

There are $n_1 n_2$ possible pairs of p and q, and that s $(1 \leq s \leq n_1 n_2)$ is unique

Thus there must exist s for any pair of p and q

(Q.E.D)

Example

$n_1 = 3, n_2 = 5$

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
q	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0

Quadratic equation for factoring

Consider the quadratic equation

$$x^2 \equiv 1 \pmod{L} \qquad \text{...(1)}$$

Here $L = n_1 n_2$ with $\gcd(n_1, n_2) = 1$

Then there exist *nontrivial solutions* such that

$$x \equiv \pm s \pmod{L}$$

Here s is in the range $1 < s < L - 1$, and the gcd of L and $s \pm 1$ gives a nontrivial factor of L

Trivial solutions

$$x = \pm 1 \pmod{L}$$

Thus $1, L-1, L$ are excluded as candidates for nontrivial solutions

Quadratic equation for factoring

Proof

Chinese remainder theorem assures there exists s $(1 < s < L - 1)$ that satisfies

$$s \equiv 1 \pmod{n_1}$$
$$s \equiv -1 \pmod{n_2}$$

$$s = 1 \Rightarrow \begin{cases} s \equiv 1 \pmod{n_1} \\ s \equiv 1 \pmod{n_2} \end{cases}$$
$$s = L - 1 \Rightarrow \begin{cases} s \equiv -1 \pmod{n_1} \\ s \equiv -1 \pmod{n_2} \end{cases}$$
$$s = L \Rightarrow \begin{cases} s \equiv 0 \pmod{n_1} \\ s \equiv 0 \pmod{n_2} \end{cases}$$

This is a nontrivial solution to Eq. (1), because

$$\begin{aligned} s^2 - 1 &\equiv 0 \pmod{n_1} \\ s^2 - 1 &\equiv 0 \pmod{n_2} \end{aligned} \quad \Rightarrow \quad s^2 - 1 \equiv 0 \pmod{L}$$

$$\gcd(n_1, n_2) = 1$$

Quadratic equation for factoring

Proof (cont'd)

Therefore,

$$(s+1)(s-1) \equiv 0 \pmod{L}$$

On the other hand,

$$0 < s-1 < s+1 < L \qquad 1 < s < L-1$$

Hence the gcd of L and $s \pm 1$ is a nontrivial factor of L, and much the same argument holds for

$$s \equiv -1 \pmod{n_1}$$

$$s \equiv 1 \pmod{n_2}$$

(Q.E.D)

Quadratic equation for factoring

Example

$n_1 = 3, n_2 = 5$

Trivial solutions

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
q	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0

Nontrivial solutions

$$\begin{cases} 4 \equiv 1 \pmod{3} \\ 4 \equiv -1 \pmod{5} \end{cases} \Rightarrow \begin{cases} \gcd(15,\ 3) = 3 \\ \gcd(15,\ 5) = 5 \end{cases}$$

$$\begin{cases} 11 \equiv -1 \pmod{3} \\ 11 \equiv 1 \pmod{5} \end{cases} \Rightarrow \begin{cases} \gcd(15,\ 10) = 5 \\ \gcd(15,\ 12) = 3 \end{cases}$$

Order of a modulo L

Definition

The least positive integer r that satisfies

$$a^r \equiv 1 \pmod{L}$$

a is in the range $0 \le a \le L-1$, and co-prime to L

Solving Eq. (1)

$$x^2 \equiv 1 \pmod{L}$$

Find r, and *if r is even*, set

$$s \equiv a^{r/2} \pmod{L}$$

If we are lucky, this is a nontrivial solution to Eq. (1), and we can factor L!

Order of a modulo $L = 15$

Factoring 15

a	r	$a^{r/2} \pm 1$	gcd w/ 15	
2	4	3, 5	3, 5	$2^4 = 16 \equiv 1$
4	2	3, 5	3, 5	$4^2 = 16 \equiv 1$
7	4	48, 50	3, 5	$7^4 = (49)^2 \equiv 4^2 \equiv 1$
8	4	63, 65	3, 5	$8^4 \equiv (-7)^4 \equiv 1$
11	2	10, 15	5, 3	$11^2 \equiv (-4)^2 \equiv 1$
13	4	168, 170	3, 5	$13^4 \equiv (-2)^4 \equiv 1$

We already know “14” yields a trivial solution, so, may well set the range of a as $1 < a < 14$

Order of a modulo $L = 21$

Factoring 21

a	r	$a^{r/2} \pm 1$	gcd w/ 21	
2	6	7, 9	7, 3	
4	3			Odd r
5	6	124, 126	19, 21	Trivial solution
8	2	7, 9	7, 3	
10	6	999, 1001	3, 7	
11	6	1330, 1332	7, 3	
13	2	12, 14	3, 7	
16	3			Odd r
17	6	4912, 4914	19, 21	Trivial solution
19	6	6858, 6860	3, 7	

"ay modulo L" is a permutation

Define $\pi(y)$ as $ay \pmod{L}$

Example

$\gcd(L, a) = 1$

$L = 15, a = 7$

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi(y)$	0	7	14	6	13	5	12	4	11	3	10	2	9	1	8

$7 \times 0 \pmod{15} = 0$

$7 \times 1 \pmod{15} = 7$

$7 \times 2 \pmod{15} = 14$

$7 \times 3 \pmod{15} = 6$

$11 \times 0 \pmod{15} = 0$

$11 \times 1 \pmod{15} = 11$

$11 \times 2 \pmod{15} = 7$

$11 \times 3 \pmod{15} = 3$

$L = 15, a = 11$

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi(y)$	0	11	7	3	14	10	6	2	13	9	5	1	12	8	4

Reduction to order finding

Now we can identify "ay mod L" as "permutation"

$$\pi(y) \Leftrightarrow ay \pmod{L}$$

For instance,

$$\pi^3(y) \Leftrightarrow a(a(ay)) \pmod{L}$$
$$\Leftrightarrow a^3 y \pmod{L}$$

Thus "finding the order of a mod L" is equivalent to "finding the order of $\pi(1)$"

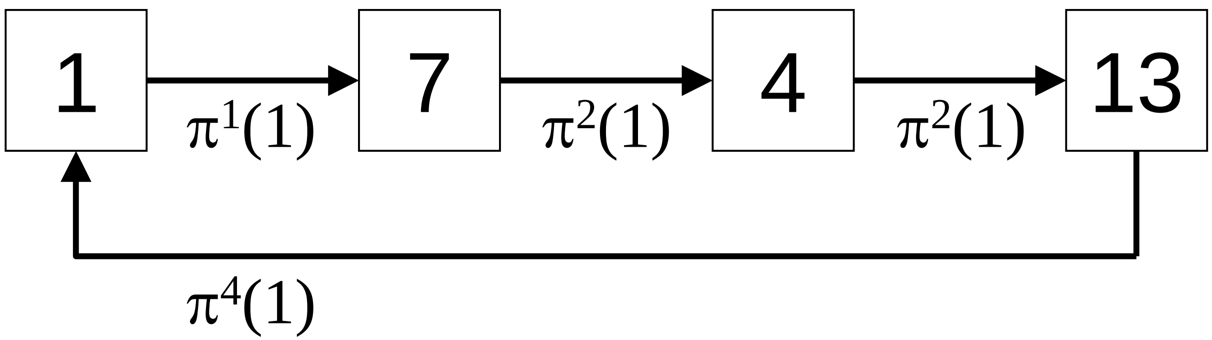
$$a^r \equiv 1 \pmod{L} \Leftrightarrow \pi^r(1) = 1$$

Order of a modulo L

Example

$L = 15,\ a = 7$

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi(y)$	0	7	14	6	13	5	12	4	11	3	10	2	9	1	8



$$7^{4/2} - 1 = 48 \quad \rightarrow \quad \gcd(15,\ 48) = 3$$

$$7^{4/2} + 1 = 50 \quad \rightarrow \quad \gcd(15,\ 50) = 5$$

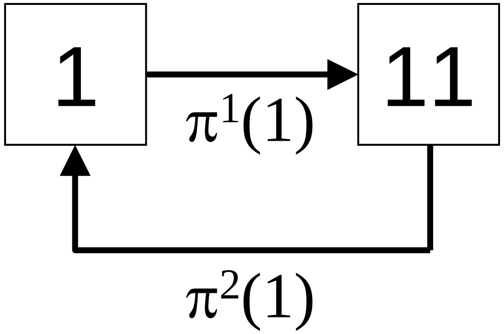
Succeed!

Order of a modulo L

Example

$L = 15,\ a = 11$

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi(y)$	0	11	7	3	14	10	6	2	13	9	5	1	12	8	4



$$11^{2/2} - 1 = 10 \quad \rightarrow \quad \gcd(15,\ 10) = 5$$

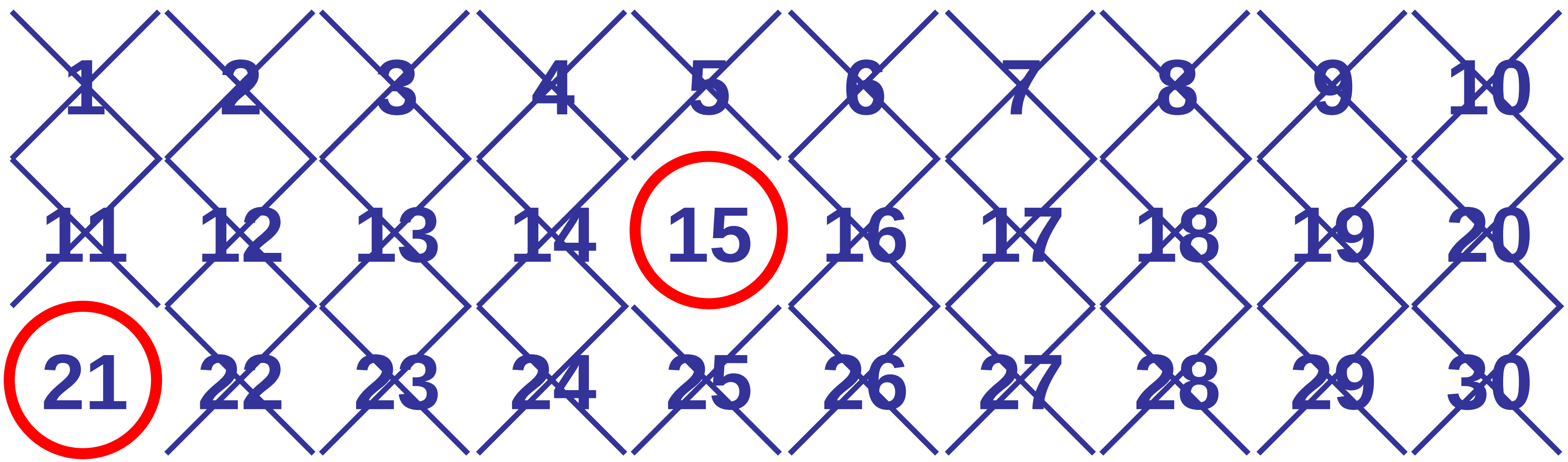
$$11^{2/2} + 1 = 12 \quad \rightarrow \quad \gcd(15,\ 12) = 3$$

Succeed!

Factoring algorithm

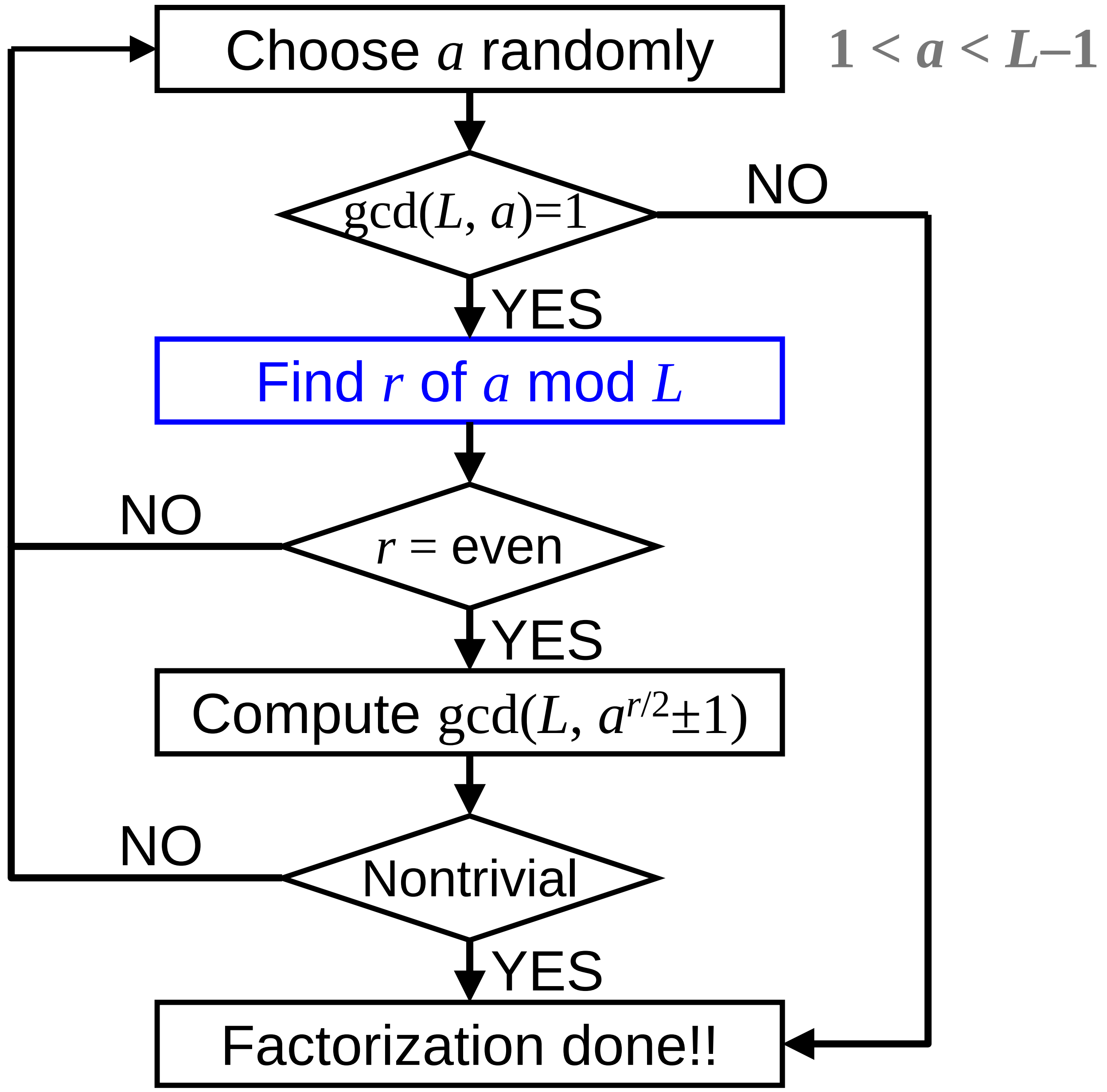
The algorithm ***fails*** when L is ...

1. even
2. a prime number
3. a prime power



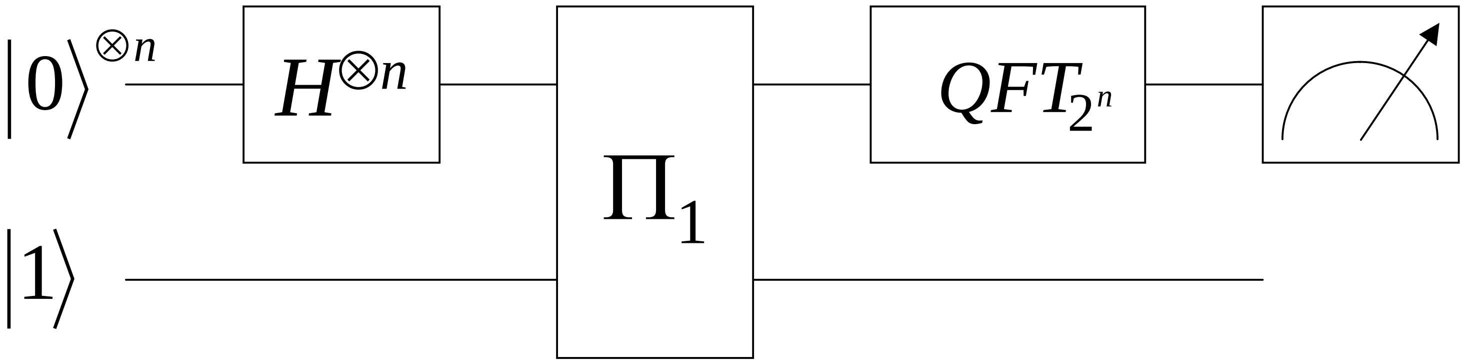
Those can be checked efficiently by classical methods before we run the algorithm

Flowchart



Choose a randomly
$1 < a < L-1$
$\gcd(L, a)=1$
NO
YES
Find r of a mod L
NO
r = even
YES
Compute $\gcd(L, a^{r/2}\pm 1)$
NO
Nontrivial
YES
Factorization done!!

Order finding for factoring



$$\Pi_1 |x\rangle|1\rangle = |x\rangle|\pi^x(1)\rangle$$
$$= |x\rangle|a^x (\mathrm{mod}\, L)\rangle$$

Nothing changes...
We have only to replace $\pi(y)$
by $ay\ (\mathrm{mod}\ L)$ with $y = 1$

Remaining issues

Now is the time to answer those questions!

- The measurement does not give us r itself, then how to obtain r out of the measurement result?
- What if r does not divide N?
- How to construct the Π_1 gate?
- If it remains a black box, how can the algorithm be useful?

Remaining issues

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Continued fractions algorithm

Split | **Invert**

$$\alpha = \frac{31}{13} = 2 + \frac{5}{13} = 2 + \frac{1}{\frac{13}{5}}$$

$$= 2 + \cfrac{1}{2 + \cfrac{3}{5}} = 2 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{3}}}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{2}{3}}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}}}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} = [2,2,1,1,2]$$

Continued fractions algorithm

$$\frac{p_0}{q_0} = [2] = \frac{2}{1} = 2$$

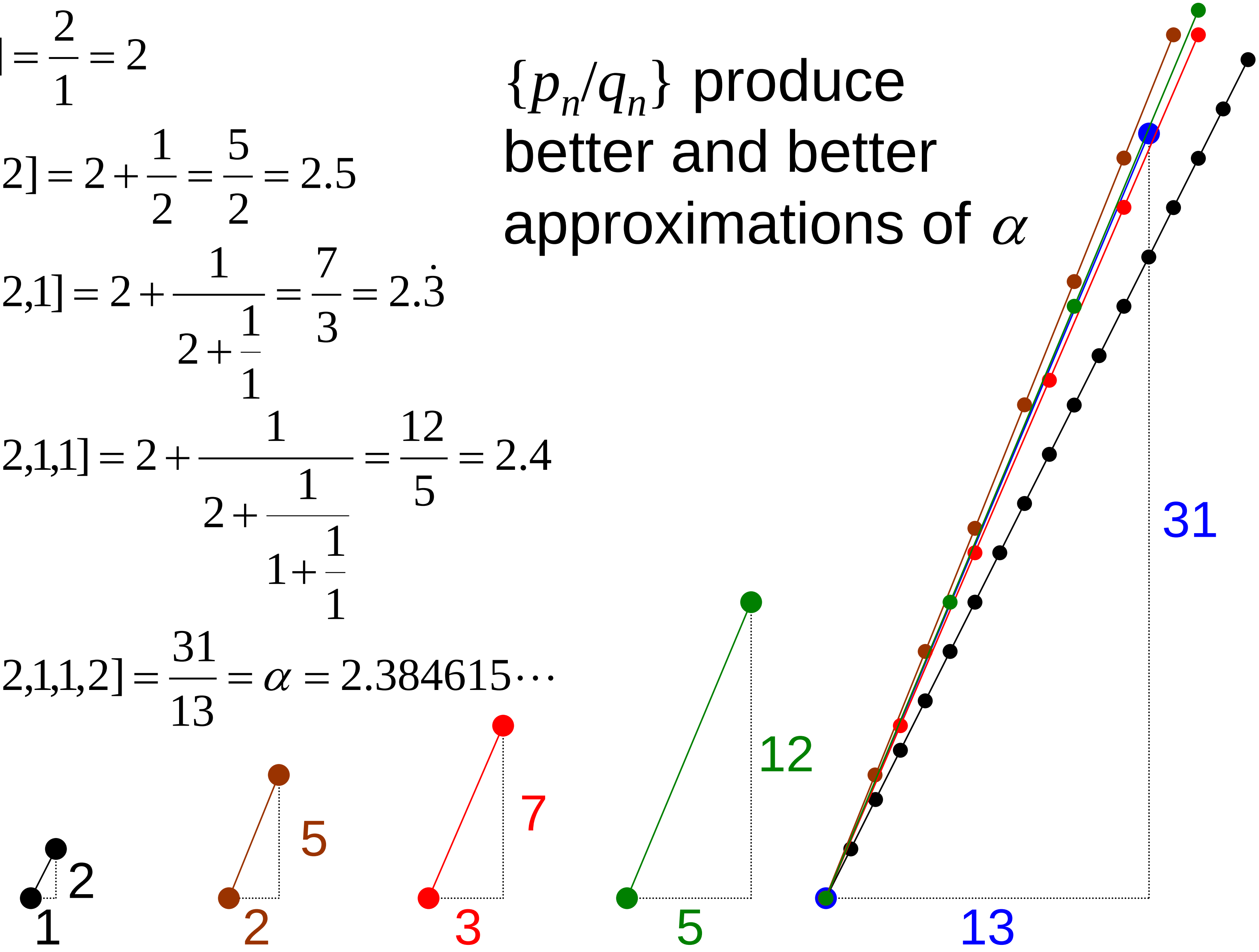
$$\frac{p_1}{q_1} = [2,2] = 2 + \frac{1}{2} = \frac{5}{2} = 2.5$$

$$\frac{p_2}{q_2} = [2,2,1] = 2 + \cfrac{1}{2 + \cfrac{1}{1}} = \frac{7}{3} = 2.\dot{3}$$

$$\frac{p_3}{q_3} = [2,2,1,1] = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{12}{5} = 2.4$$

$$\frac{p_4}{q_4} = [2,2,1,1,2] = \frac{31}{13} = \alpha = 2.384615\cdots$$

$\{p_n/q_n\}$ produce better and better approximations of α



Continued fractions algorithm

Given the continued fraction expansion

$$\alpha = [a_0, a_1, \cdots, a_m]$$

Then the nth convergent of α is given by

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

with

$$(p_{-2}, q_{-2}) = (0,1)$$
$$(p_{-1}, q_{-1}) = (1,0)$$

n	–2	–1	0	1	2	3	4
a_n	–	–	2	2	1	1	2
p_n	0	1	**2**	**5**	**7**	**12**	**31**
q_n	1	0	**1**	**2**	**3**	**5**	**13**

It can be shown that p_n and q_n are co-prime

Continued fractions algorithm

Suppose k/r is a rational number such that

$$\left|\frac{k}{r} - \varphi\right| \le \frac{1}{2r^2}$$

Then k/r is a convergent of the continued fraction for φ

The inequality holds if φ is an approximation of k/r accurate to $2l + 1$ bits

$$\left|\frac{k}{r} - \varphi\right| \le \frac{1}{2^{2l+1}} \le \frac{1}{2r^2}$$

$$l \equiv \lceil \log_2 L \rceil \quad (2^{l-1} < L \le 2^l)$$

$$2^{2l+1} = 2(2^l)^2 \ge 2L^2 \ge 2r^2$$

Case study: Factoring 39

Step 1: Choose random a coprime to L

$$a = 7$$

Step 2: Find r

$$r = 12$$

Continued fractions algorithm after measurement

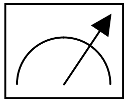
Step 3: Compute $\gcd(L,\ a^{r/2} \pm 1)$

$$7^{12/2} - 1 \equiv 24 \pmod{39} \quad \rightarrow \quad \gcd(39,\ 24) = 3$$

$$7^{12/2} + 1 \equiv 26 \pmod{39} \quad \rightarrow \quad \gcd(39,\ 26) = 13$$

Determining r after measurement

$$\approx \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{2\pi i m k / r} \left| \frac{N}{r} k \right\rangle$$



$$\longrightarrow \quad |\lambda\rangle \approx \left| \frac{N}{r} k \right\rangle$$

<u>Example</u>

$$L = 39$$

$$a = 7$$

$$r = 12$$

$$l = \lceil \log_2 L \rceil = 6$$

$$N = 2^{2l+1} = 8192$$

$$k = 5$$

$$\lambda = 3413$$

$$\frac{Nk}{r} = \frac{8192 \cdot 5}{12} = 3413.\dot{3}$$

$$\frac{\lambda}{N} \approx \frac{k}{r}$$

$$\frac{3413}{8192} = 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{170 + \cfrac{1}{4}}}}}$$

$$= [0,2,2,2,170,4]$$

Determining r after measurement

n	–2	–1	0	1	2	3	4	5
a_n	–	–	0	2	2	2	170	4
p_n	0	1	**0**	**1**	**2**	**5**	**852**	**3413**
q_n	1	0	**1**	**2**	**5**	**12**	**2045**	**8192**

$$\underbrace{\frac{p_1}{q_1}=\frac{1}{2} \quad \frac{p_2}{q_2}=\frac{2}{5} \quad \frac{p_3}{q_3}=\frac{5}{12}} \quad \underbrace{\frac{p_4}{q_4}=\frac{852}{2045} \quad \frac{p_5}{q_5}=\frac{3413}{8192}}$$

Candidates for k/r

$r \le L = 39$

Compute $a^{q_n} \pmod{L}$

Know that $q_3 = 12$ is the order

Remaining issues

Now is the time to answer those questions!

- The measurement does not give us r itself, then how to obtain r out of the measurement result?
- What if r does not divide N?
- How to construct the Π_1 gate?
- If it remains a black box, how can the algorithm be useful?

Π_1 gate

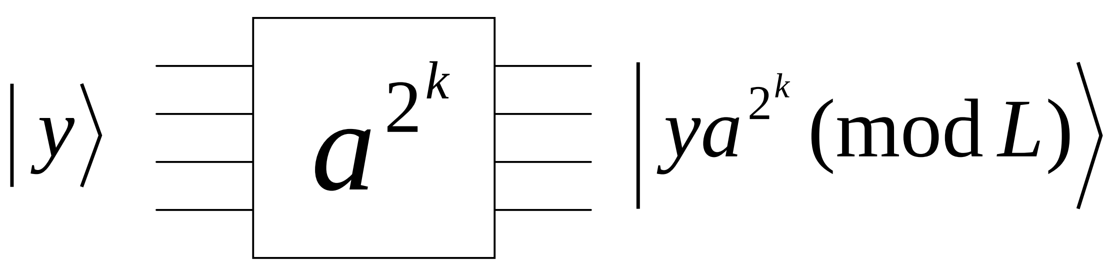
$$\Pi_1 |x\rangle|1\rangle = |x\rangle\left|a^x (\mathrm{mod}\, L)\right\rangle$$

$$x = 2^{n-1}x_n + 2^{n-2}x_{n-1} + \cdots + 2x_1 + x_0$$

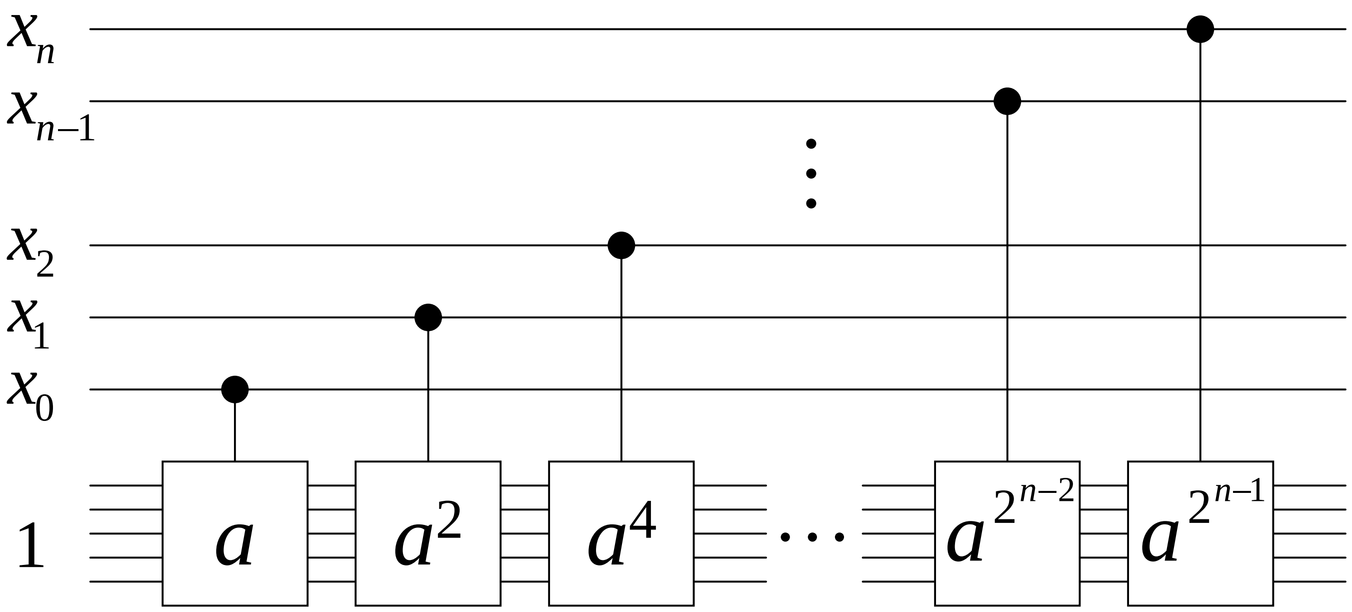
$$a^x (\mathrm{mod}\, L) = a^{2^{n-1}x_n + 2^{n-2}x_{n-1} + \cdots 2x_1 + x_0} (\mathrm{mod}\, L)$$

$$= \underbrace{[a^{2^{n-1}} (\mathrm{mod}\, L)]^{x_n} [a^{2^{n-2}} (\mathrm{mod}\, L)]^{x_{n-1}} \cdots [a \ (\mathrm{mod}\, L)]^{x_0}}_{\text{Controlled-}U\text{ gates}}$$

Controlled-U gates



Modular exponentiation



We must at least calculate $a^{2^k} \pmod{L}$ classically by repeated squaring

$$(a^{2^{k-1}})^2 = a^{2^k}$$

The circuit is constructed without knowing the order itself

Case study: Factoring 15

<u>Step 1: Choose random a coprime to L</u>

$$a = 7$$

<u>Step 2: Find r</u>

$$r = 4$$

Concrete construction of Π gate due to Vandersypen *et al*. will be given in the following slides

<u>Step 3: Compute $\gcd(L, a^{r/2} \pm 1)$</u>

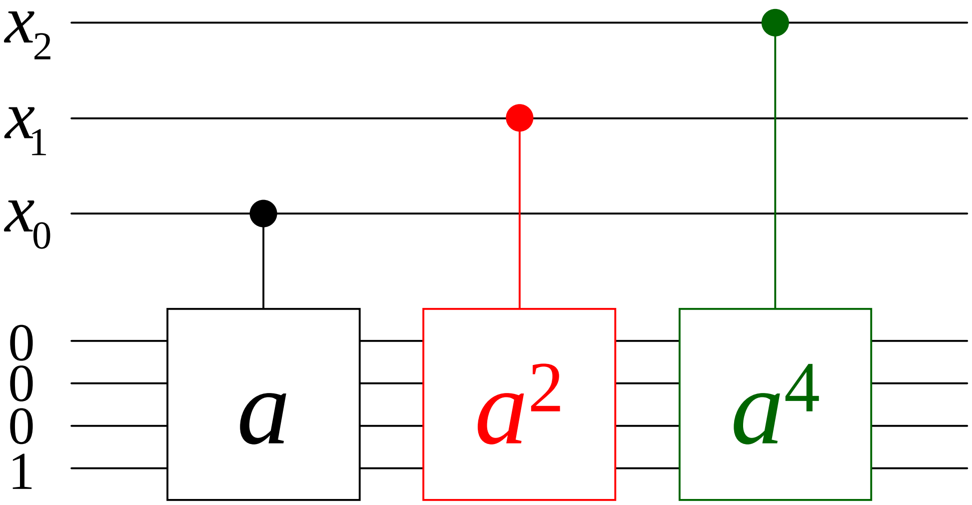
$$7^{4/2} - 1 = 48 \quad \rightarrow \quad \gcd(15,\ 48) = 3$$

$$7^{4/2} + 1 = 50 \quad \rightarrow \quad \gcd(15,\ 50) = 5$$

Finding r of a modulo 15

$$a^{2^k}\left|y\right\rangle = \left|ya^{2^k}\,(\mathrm{mod}\,L)\right\rangle$$

a	a^2	a^4	r
2	4	1	4
4	1	1	2
7	4	1	4
8	4	1	4
11	1	1	2
13	4	1	4



Only needed when $a = 2, 7, 8, 13$

Trivial

In reality, if $r = 2^k$, a quantum computer is not necessary (Know r during repeated squaring)

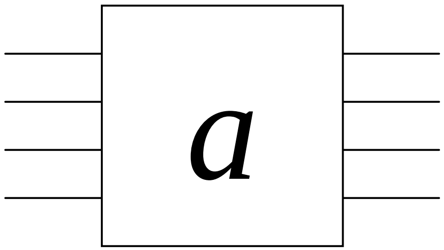
Modular exponentiation

Example: $a = 7$

$$
\begin{aligned}
& a \pmod{15} \\
= \quad & (a-1)+1 \pmod{15} \\
= \quad & (4\cdot 1 + 2\cdot 1)+1 \pmod{15}
\end{aligned}
$$

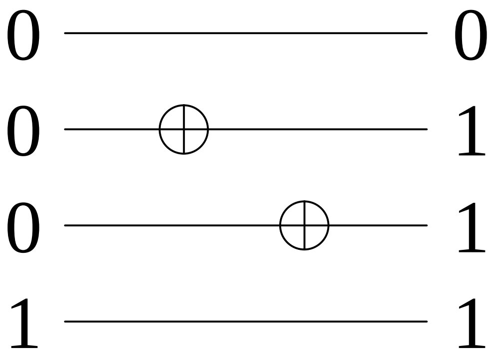
$$y = 8y_3 + 4y_2 + 2y_1 + y_0$$

$|y_3 y_2 y_1 y_0\rangle$
$= |0001\rangle$



$|0111\rangle$

$=$



For other a, the gate is constructed in a similar fashion

Modular exponentiation

Example: $a = 7$ (and 2, 8, 13)

$$a^2 y \pmod{15}$$

$$= 4 \times (8y_3 + 4y_2 + 2y_1 + y_0) \pmod{15}$$

$$= 32y_3 + 16y_2 + 8y_1 + 4y_0 \pmod{15}$$

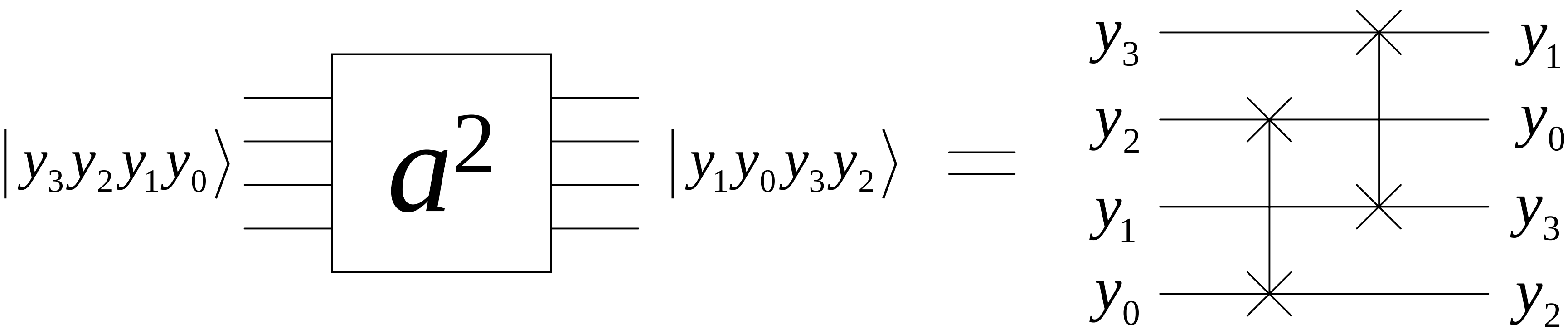
$$= 2y_3 + y_2 + 8y_1 + 4y_0 \pmod{15}$$

$$= 8y_1 + 4y_0 + 2y_3 + y_2 \pmod{15}$$

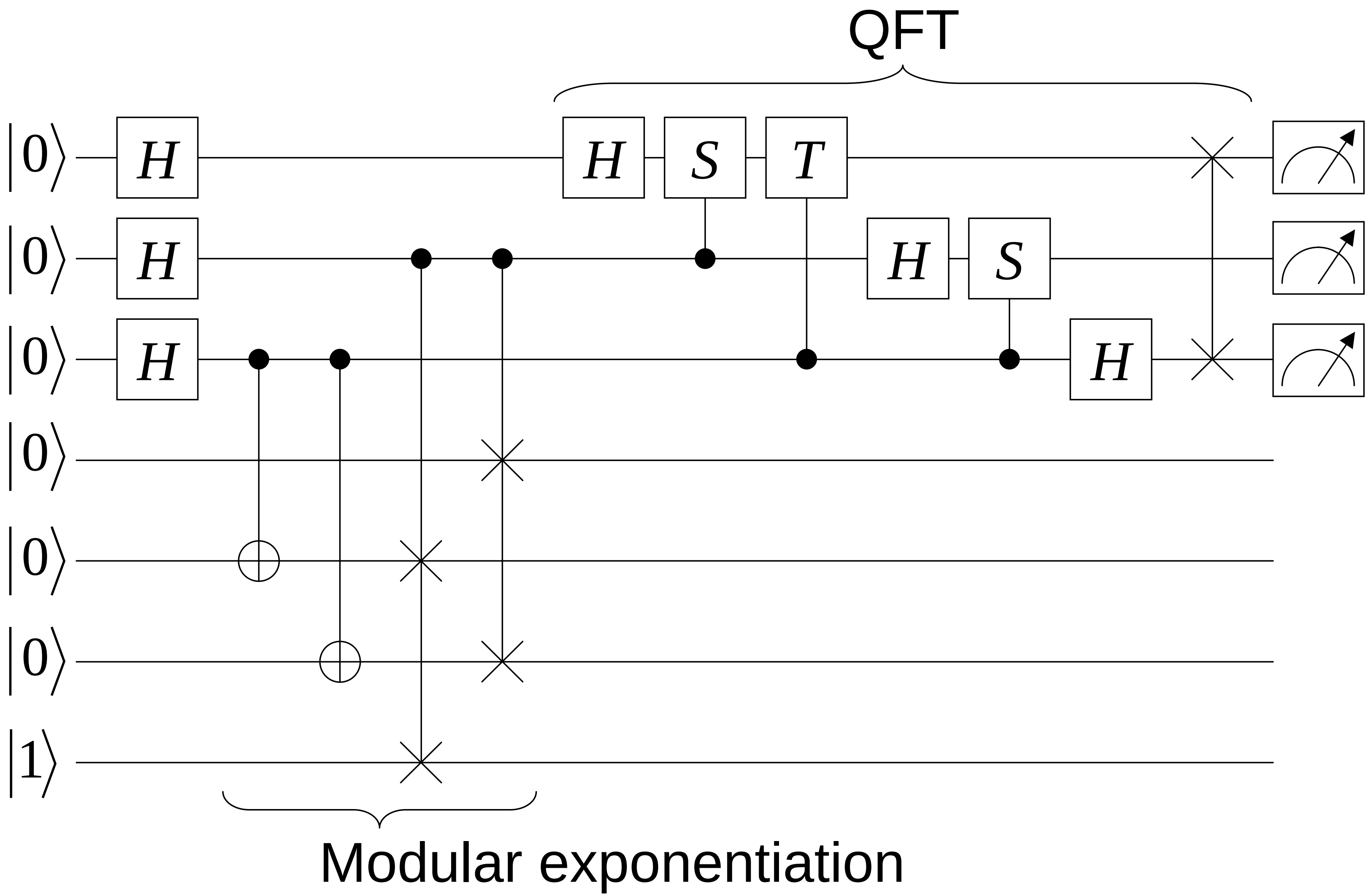
$$a^2 = 4 \pmod{15}$$

$$32 = 2 \pmod{15}$$

$$16 = 1 \pmod{15}$$

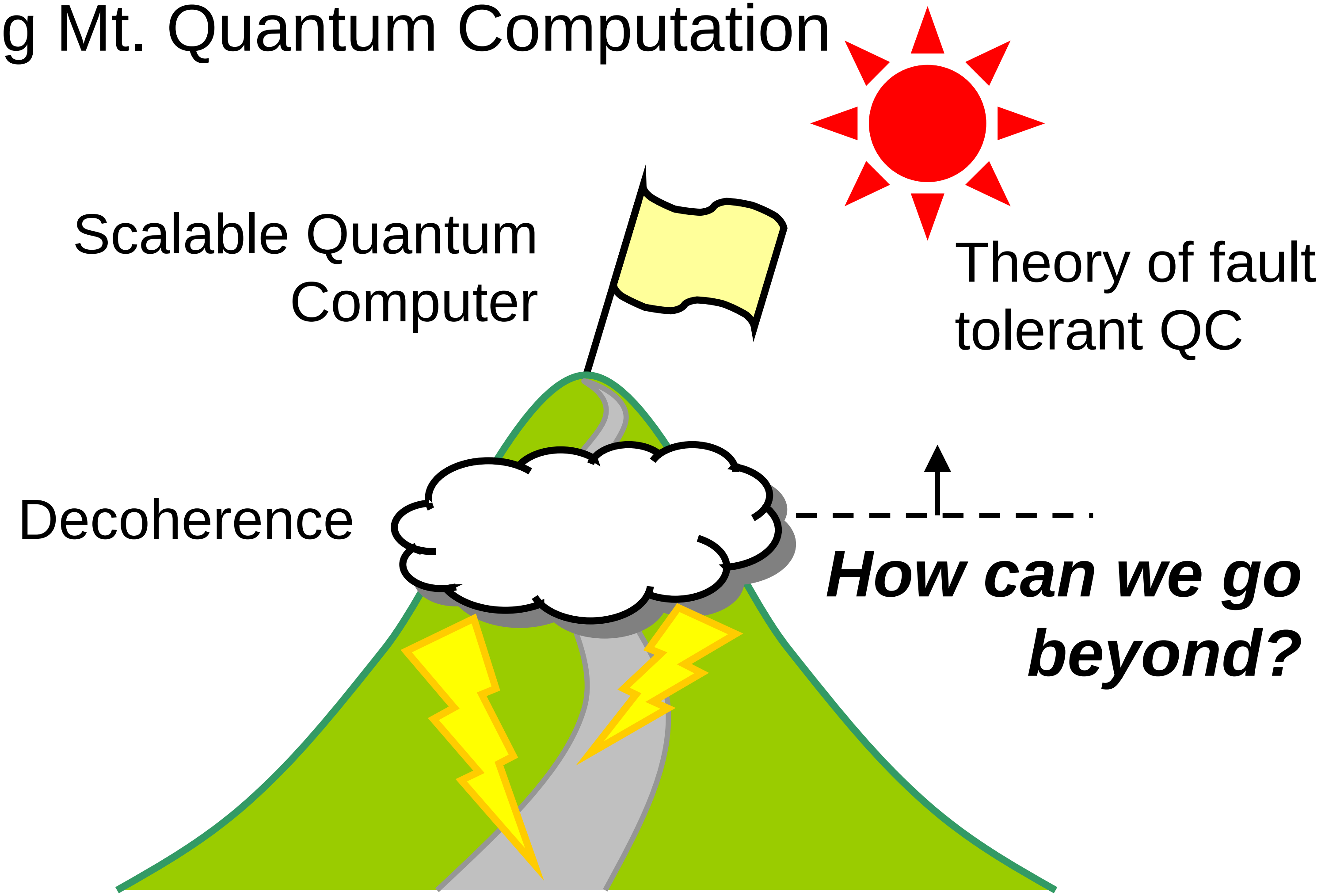


Quantum circuit for factoring 15



Where are we now?

Climbing Mt. Quantum Computation



We are still at the foot of the mountain…

Thank you for your attention!!