Grover's algorithm

School on Quantum Computing @Yagami

Day 1, Lesson 4

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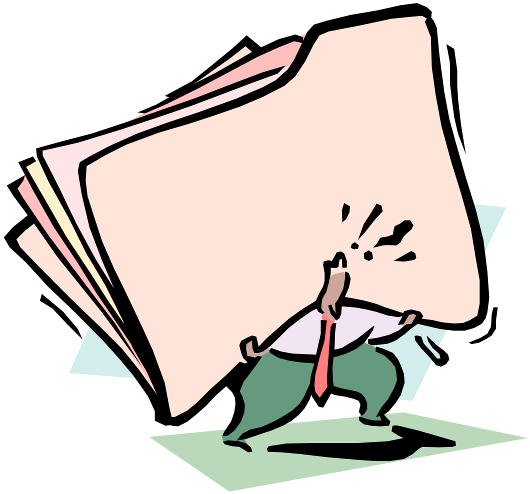
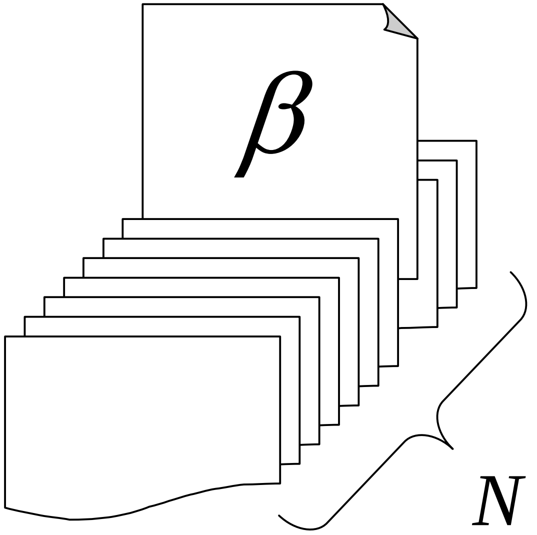
Outline

- Grover's algorithm
  - Oracle
  - Grover operator
  - Examples
    - 2-bit
    - 3bit
  - Geometric visualization
  - Performance and optimality

Database searching

Find the desired file indexed as “β”
among $N = 2^n$ files



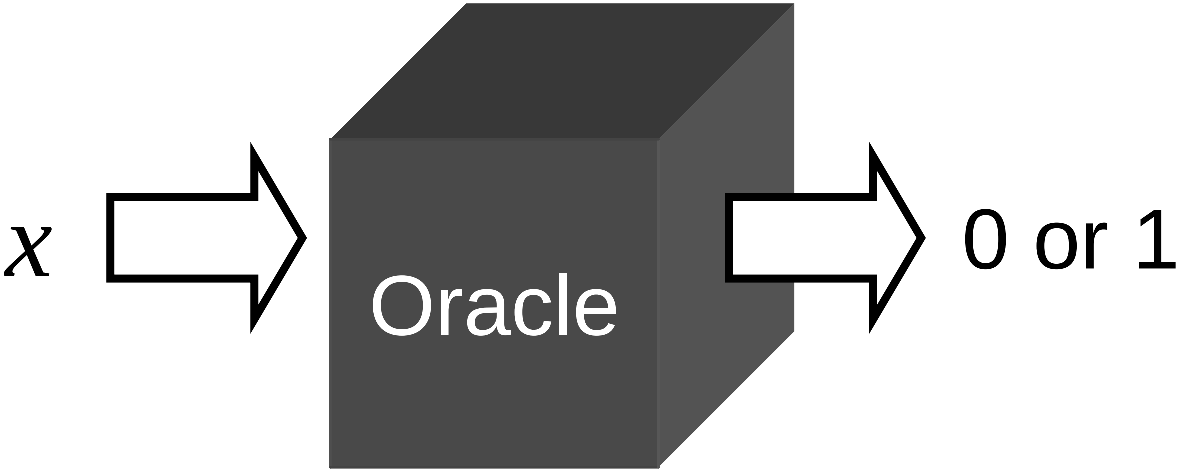


$N = 2^n$ files

Oracle

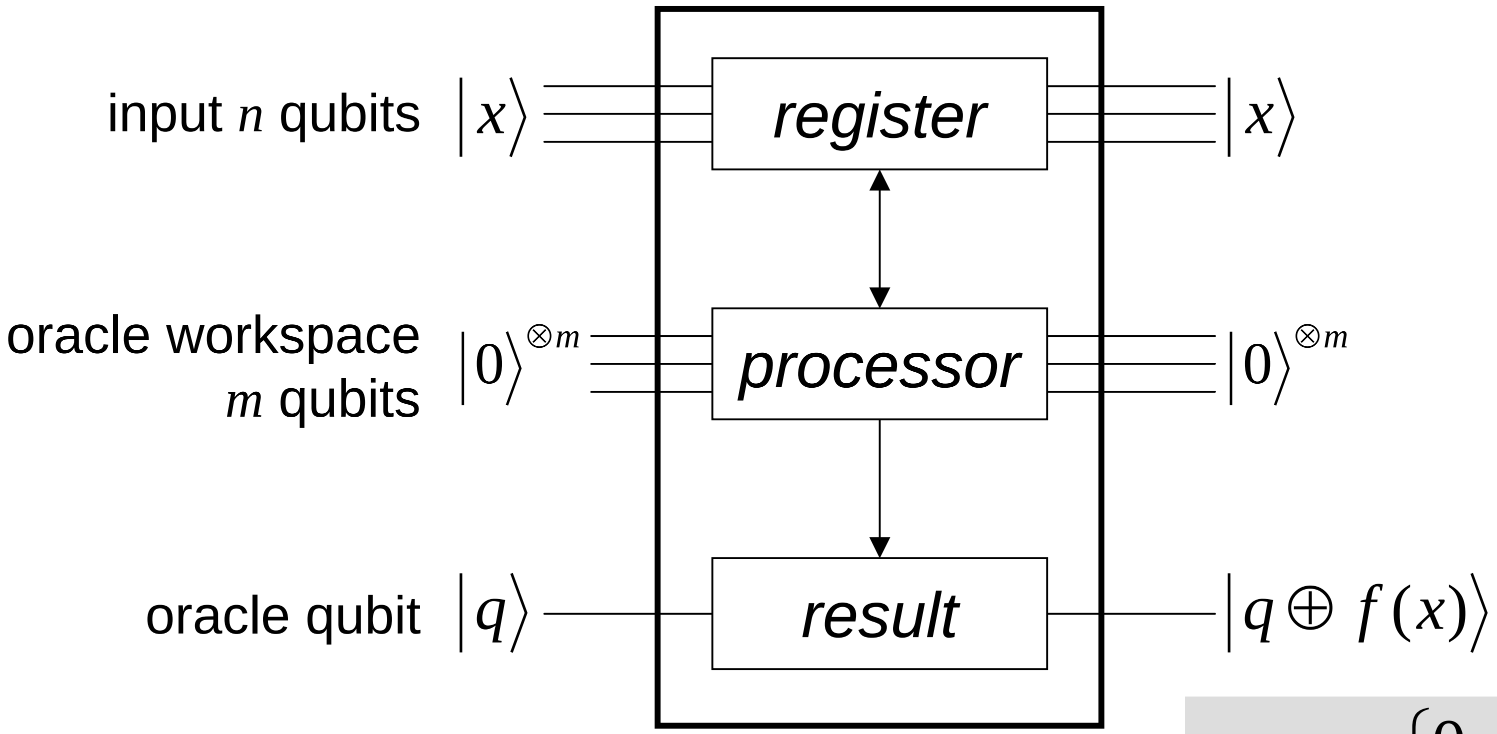
A black box that can ***recognize*** the solution, whose internal working is represented by a binary function $f(x)$

$$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$



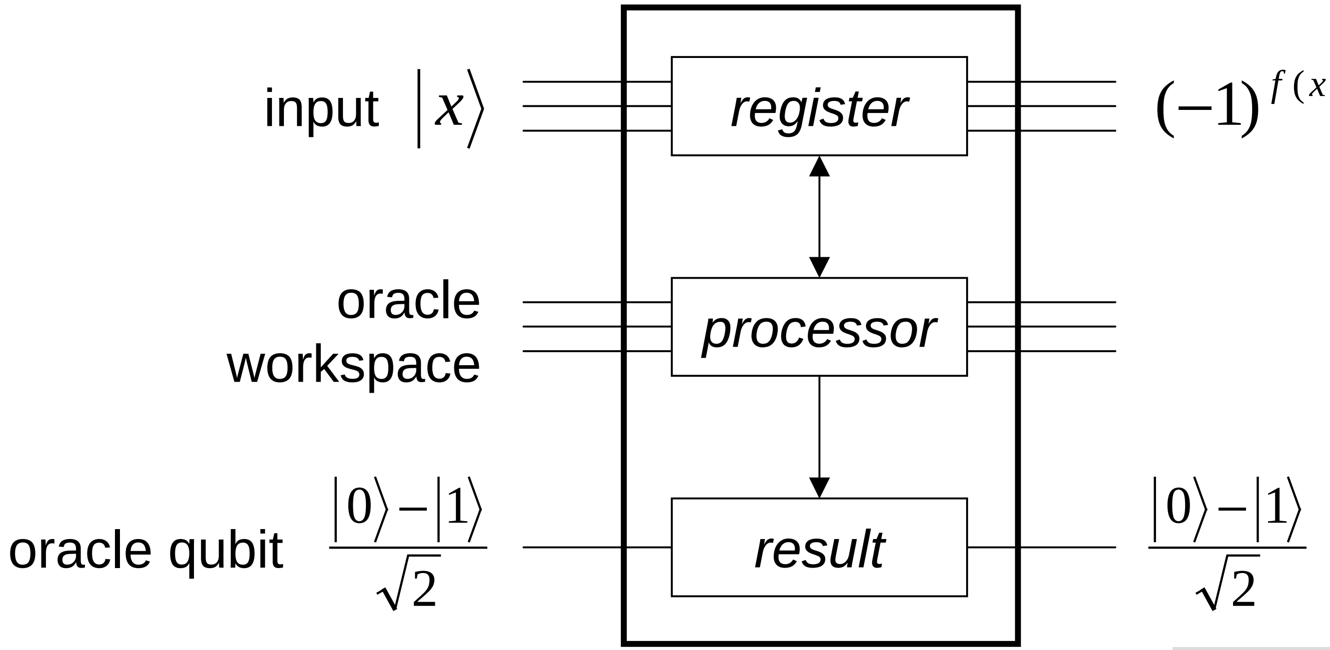
Oracle

$$|x\rangle|q\rangle \xrightarrow{O} |x\rangle|q \oplus f(x)\rangle$$



$$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$

Oracle



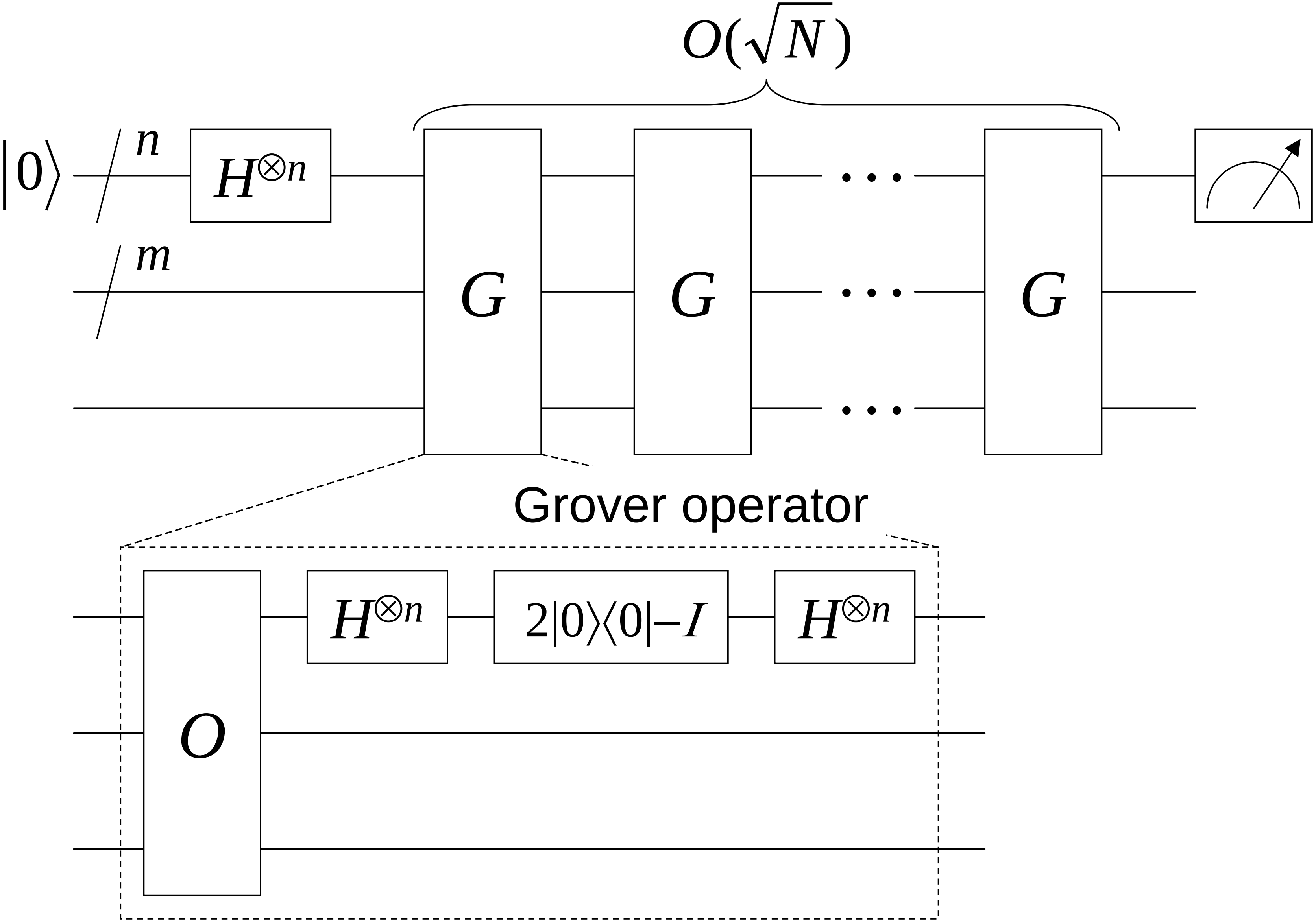
$$|x\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\xrightarrow{O}(-1)^{f(x)}|x\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$\begin{aligned}&|0\oplus f(x)\rangle-|1\oplus f(x)\rangle\\&=\begin{cases}|0\rangle-|1\rangle & \text{if } f(x)=0\\ |1\rangle-|0\rangle & \text{if } f(x)=1\end{cases}\\&=(-1)^{f(x)}\left(|0\rangle-|1\rangle\right)\end{aligned}$$

Simplify

$$|x\rangle\xrightarrow{O}(-1)^{f(x)}|x\rangle$$

Quantum search algorithm

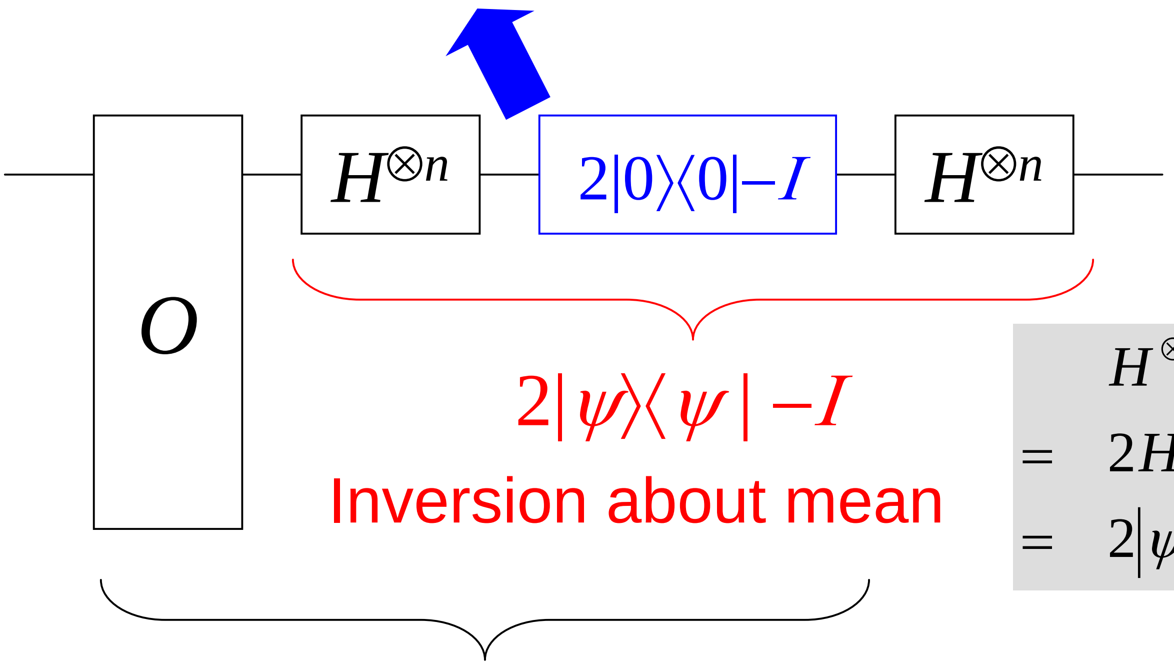


Grover operator

Phase shift operator

$$|0\rangle \rightarrow |0\rangle$$
$$|x\rangle \rightarrow -|x\rangle \quad (x \neq 0)$$

$$(2|0\rangle\langle 0| - I)|0\rangle = 2|0\rangle\langle 0|0\rangle - |0\rangle = |0\rangle$$
$$(2|0\rangle\langle 0| - I)|x\rangle = 2|0\rangle\langle 0|x\rangle - |x\rangle = -|x\rangle$$



$$\begin{aligned} & H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} \\ = \; & 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - H^{\otimes n}H^{\otimes n} \\ = \; & 2|\psi\rangle\langle\psi| - I \end{aligned}$$

$$|\psi\rangle = \frac{1}{N^{1/2}} \sum_{x=0}^{N-1} |x\rangle$$

$$G = (2|\psi\rangle\langle\psi| - I)O$$

Grover operator

$$(2|\psi\rangle\langle\psi| - I)\sum_k \alpha_k |k\rangle$$

$$= 2N^{-1}\sum_{k,k',k''} \alpha_k |k'\rangle\langle k''|k\rangle - \sum_k \alpha_k |k\rangle$$

$$= 2N^{-1}\sum_{k,k'} \alpha_k |k'\rangle - \sum_k \alpha_k |k\rangle$$

$$= \sum_k \left(2\langle\alpha\rangle - \alpha_k\right)|k\rangle$$

$$|\psi\rangle = N^{-1/2}\sum |k'\rangle$$

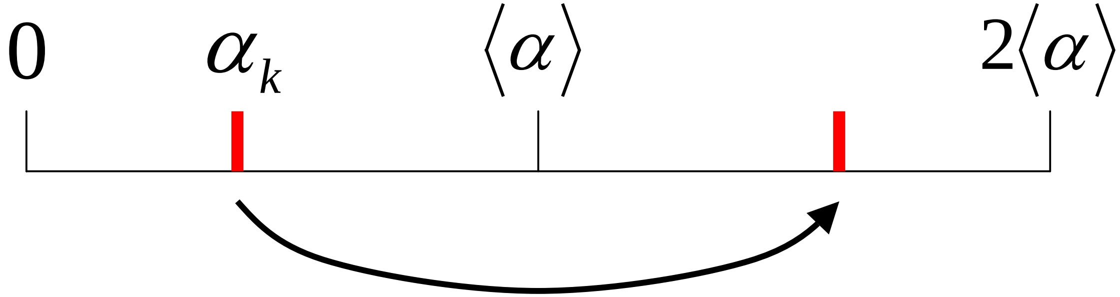
$$\langle\psi| = N^{-1/2}\sum \langle k''|$$

$$\langle k''|k\rangle = \delta_{kk''}$$

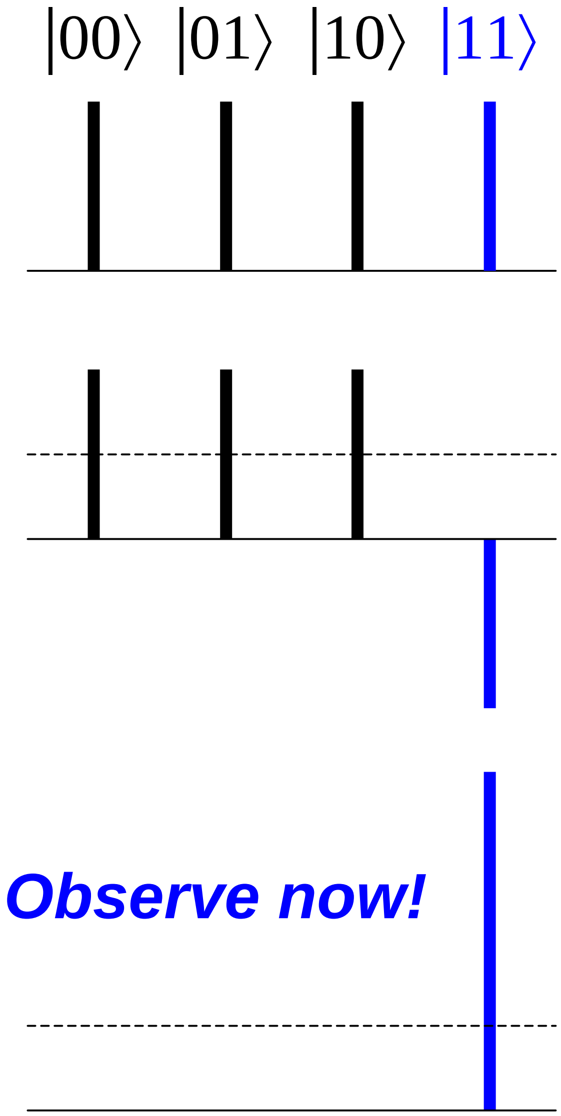
$$\langle\alpha\rangle = N^{-1}\sum \alpha_k$$

$$k' \to k$$

Inversion about mean



2-bit Grover



$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \underline{\frac{1}{2}|11\rangle}$$

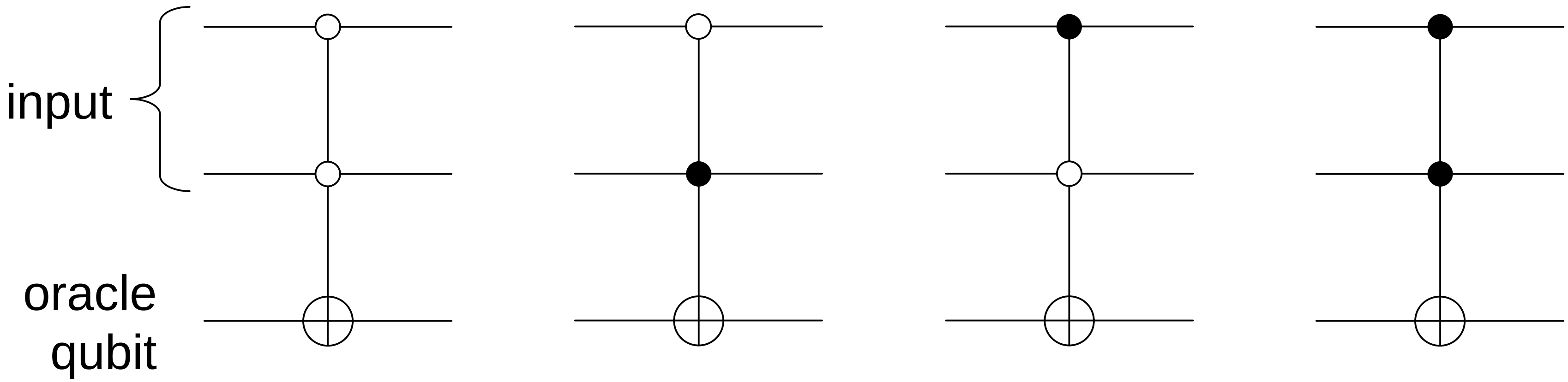
$$O|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle \underline{- \frac{1}{2}|11\rangle}$$

$$G|\psi\rangle = (2|\psi\rangle\langle\psi| - I)O|\psi\rangle = \underline{|11\rangle}$$

$$\langle\alpha\rangle = \frac{1}{4}\left(3\times\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{4}$$

$$2\langle\alpha\rangle - \alpha_k = 2\frac{1}{4} \pm \frac{1}{2}$$

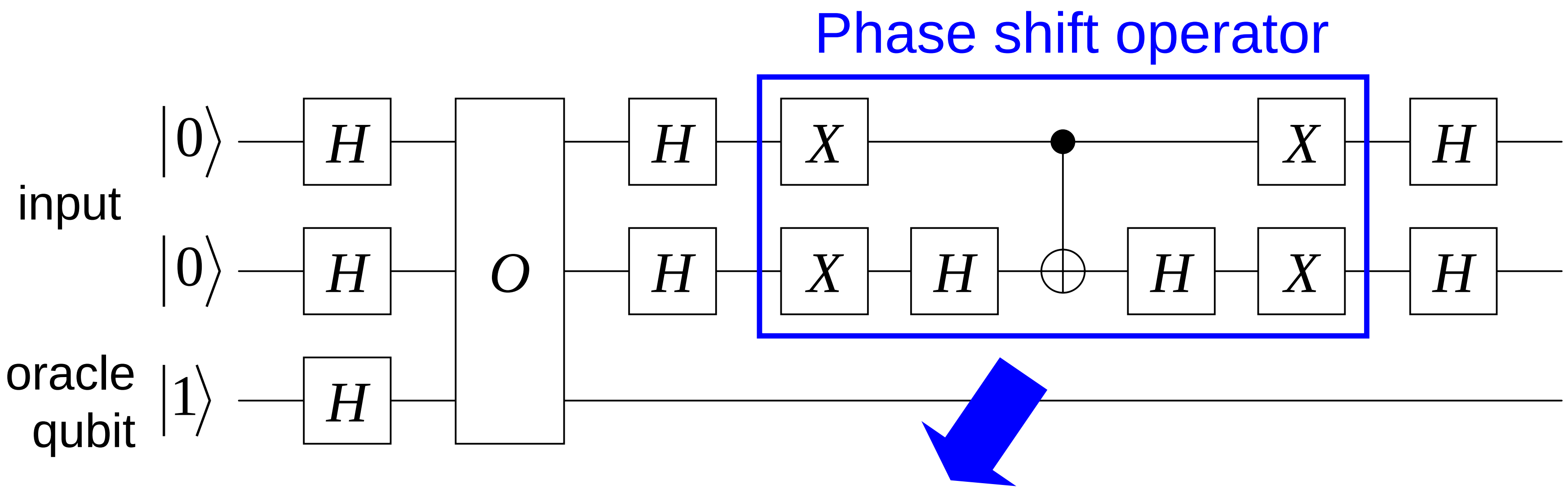
Oracle for 2-bit Grover



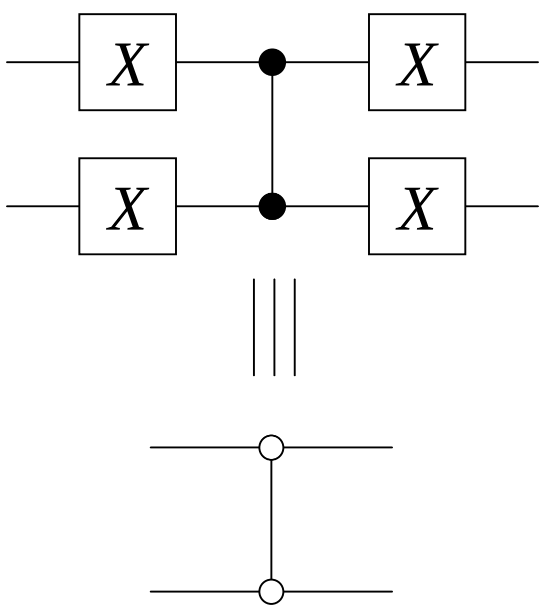
$$\beta = 0 \qquad \beta = 1 \qquad \beta = 2 \qquad \beta = 3$$

$$|x\rangle|q\rangle \xrightarrow{O} |x\rangle|q \oplus f(x)\rangle \qquad f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$

Quantum circuit for 2-bit Grover

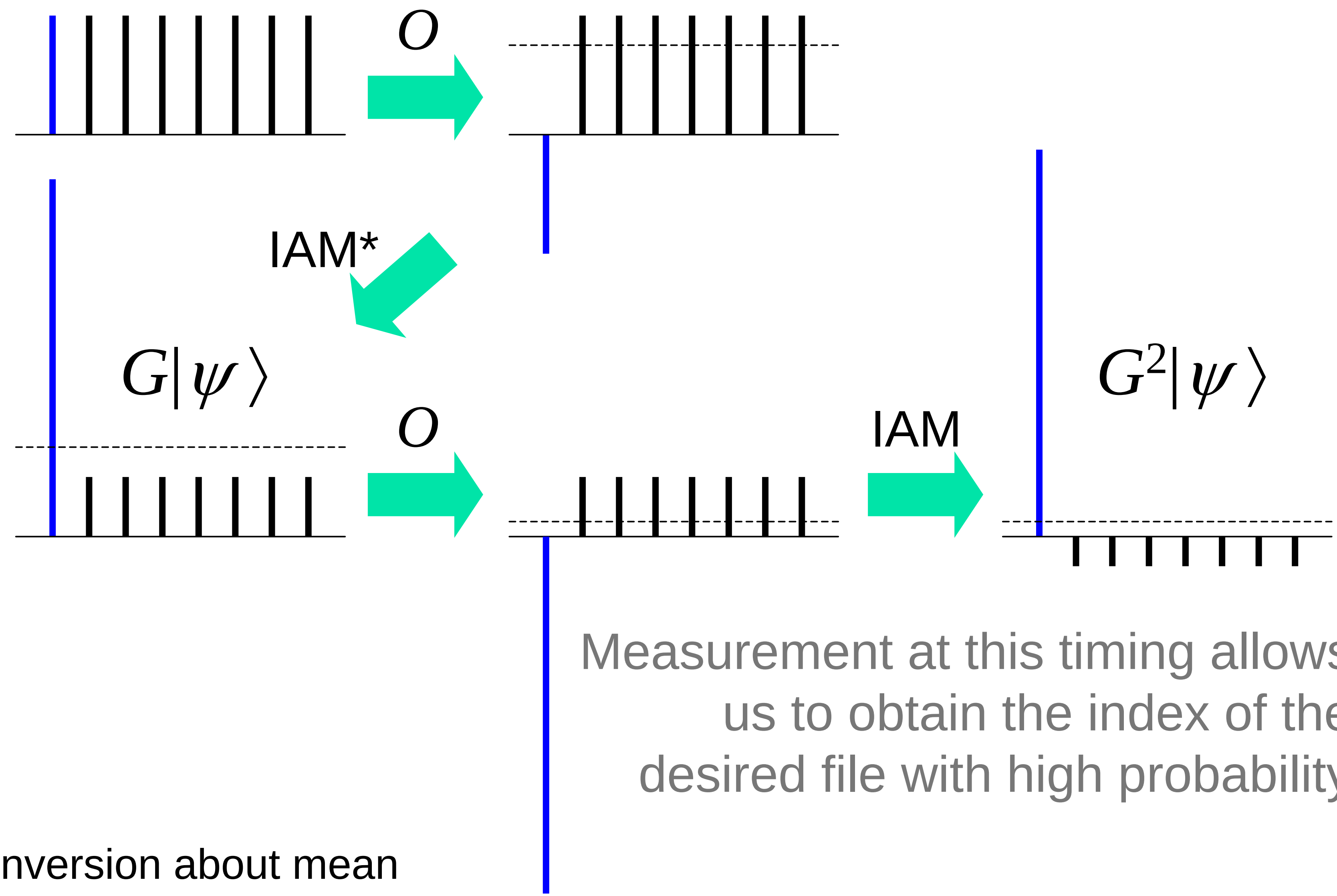


$$-(2|00\rangle\langle 00| - I) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{aligned} & X_2 X_1 CZ X_2 X_1 |a\rangle|b\rangle \\ = \; & X_2 X_1 CZ |\bar{a}\rangle|\bar{b}\rangle \\ = \; & X_2 X_1 (-1)^{\bar{a}\cdot\bar{b}} |\bar{a}\rangle|\bar{b}\rangle \\ = \; & (-1)^{\bar{a}\cdot\bar{b}} |a\rangle|b\rangle \end{aligned}$$

3-bit Grover



Measurement at this timing allows us to obtain the index of the desired file with high probability

* Inversion about mean

Geometric visualization

Sum over all x except β

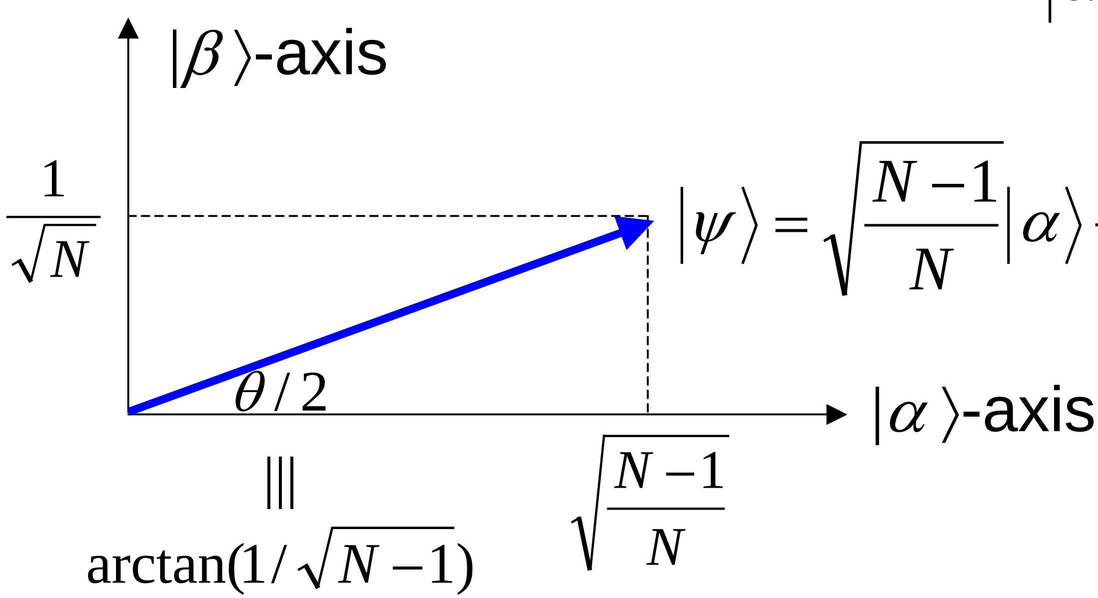
$$|\alpha\rangle \equiv \frac{1}{\sqrt{N-1}} \sum_{x}{}' |x\rangle$$

$$\langle\alpha|\alpha\rangle = \frac{1}{N-1}\sum_{x,x'}{}' \langle x'|x\rangle = 1$$

$$\langle\alpha|\beta\rangle = 0$$

The initial state $|\psi\rangle$ is visualized as a vector in the real 2D plane spanned by $|\alpha\rangle$ and $|\beta\rangle$

$$|\alpha\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\beta\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$|\psi\rangle = \sqrt{\frac{N-1}{N}}|\alpha\rangle + \frac{1}{\sqrt{N}}|\beta\rangle = \begin{pmatrix} \cos[\theta/2] \\ \sin[\theta/2] \end{pmatrix}$$

Geometric visualization

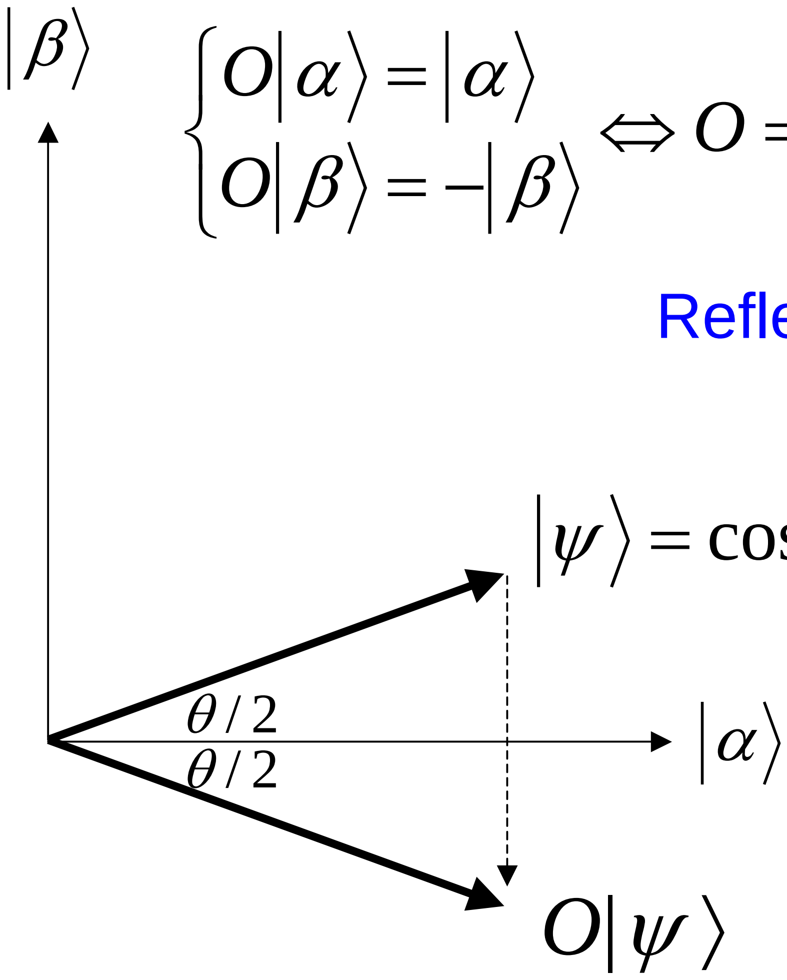
$$\left\{\begin{array}{l} O|\alpha\rangle = |\alpha\rangle \\ O|\beta\rangle = -|\beta\rangle \end{array}\right. \Leftrightarrow O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|x\rangle \xrightarrow{O} (-1)^{f(x)}|x\rangle$$

$$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$

Reflection about the $|\alpha\rangle$-axis

$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$



Geometric visualization

$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$

$$\begin{aligned}(2|\psi\rangle\langle\psi| - I)|\alpha\rangle &= 2\cos\frac{\theta}{2}|\psi\rangle - |\alpha\rangle \\ &= \left(2\cos^2\frac{\theta}{2} - 1\right)|\alpha\rangle + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|\beta\rangle \\ &= \cos\theta|\alpha\rangle + \sin\theta|\beta\rangle\end{aligned}$$

$$\begin{aligned}(2|\psi\rangle\langle\psi| - I)|\beta\rangle &= 2\sin\frac{\theta}{2}|\psi\rangle - |\beta\rangle \\ &= 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|\alpha\rangle + \left(2\sin^2\frac{\theta}{2} - 1\right)|\beta\rangle \\ &= \sin\theta|\alpha\rangle - \cos\theta|\beta\rangle\end{aligned}$$

$$\Rightarrow \quad 2|\psi\rangle\langle\psi| - I = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

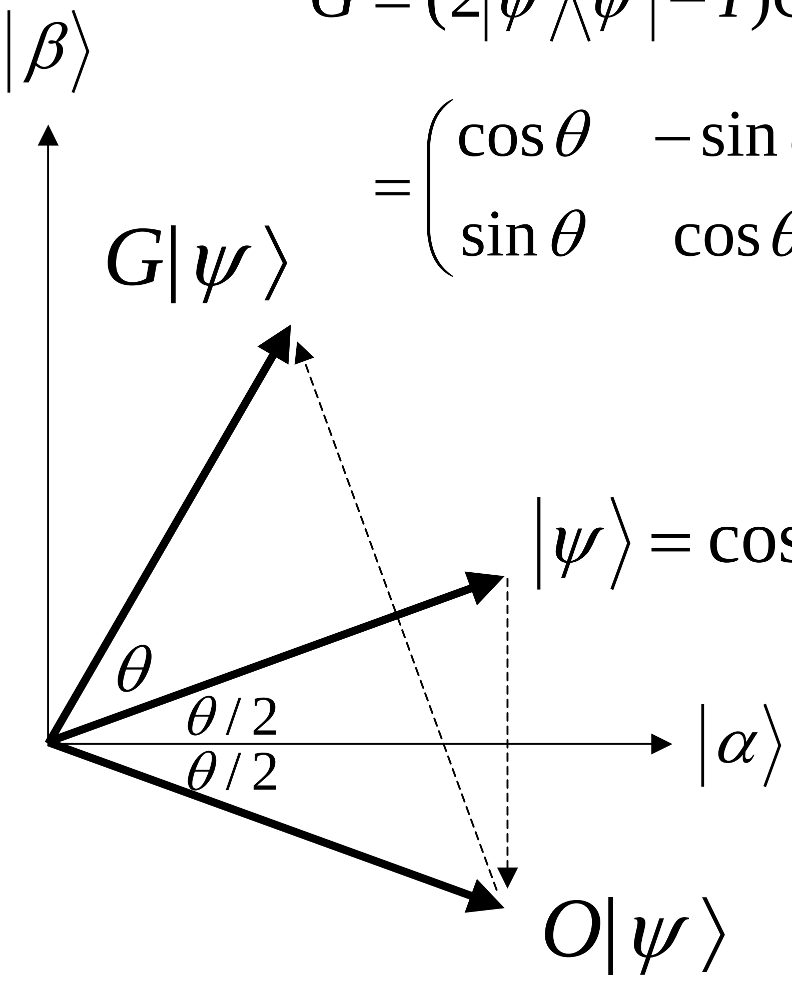
Reflection about the vector $|\psi\rangle$

Geometric visualization

$$G = (2|\psi\rangle\langle\psi| - I)O = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The product of two reflections is a rotation

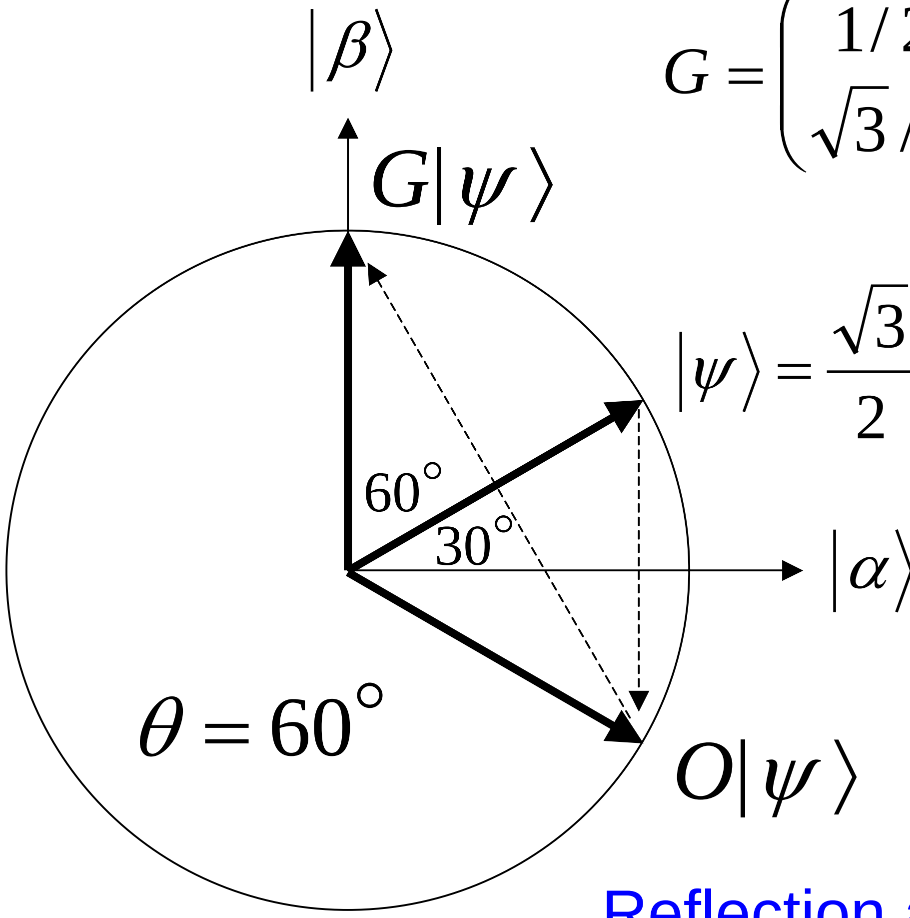


$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$

2-bit Grover

Rotation by 60 degrees

$$G = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

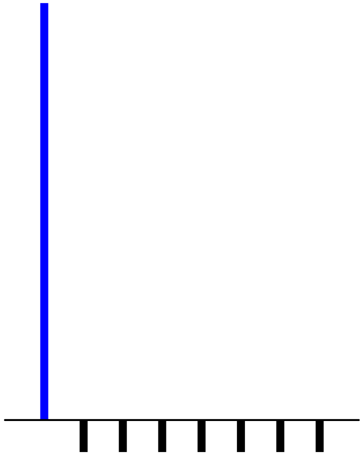


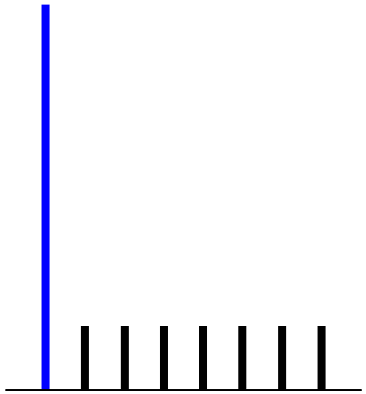
$$|\psi\rangle = \frac{\sqrt{3}}{2}|\alpha\rangle + \frac{1}{2}|\beta\rangle$$

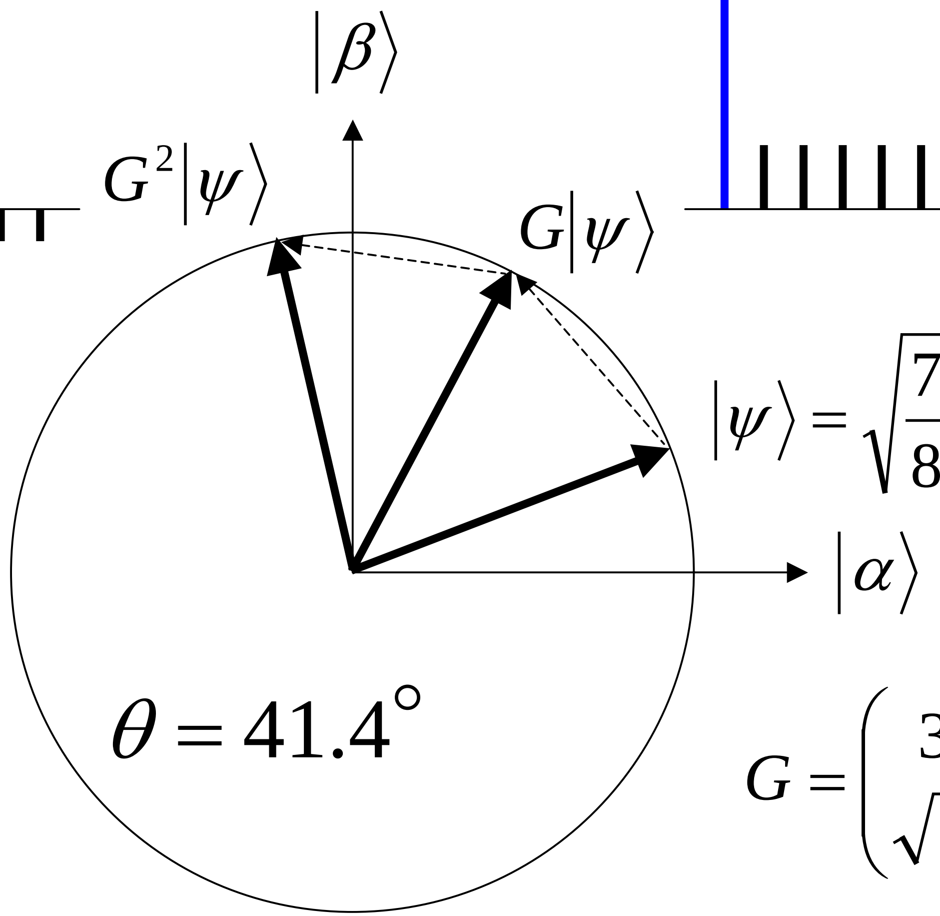
$$O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflection about the $|\alpha\rangle$-axis

3-bit Grover







$$|\psi\rangle = \sqrt{\frac{7}{8}}|\alpha\rangle + \frac{1}{\sqrt{8}}|\beta\rangle$$

$$G = \begin{pmatrix} 3/4 & -\sqrt{7}/4 \\ \sqrt{7}/4 & 3/4 \end{pmatrix}$$

Performance

The state after repeating the Grover iteration k times

$$G^k|\psi\rangle = \cos\left(\frac{2k+1}{2}\theta\right)|\alpha\rangle + \sin\left(\frac{2k+1}{2}\theta\right)|\beta\rangle$$

We terminate the iteration when

$$\frac{2k'+1}{2}\theta \approx \frac{\pi}{2}$$

Assume θ is small (i.e., N large), then

$$\sin\frac{\theta}{2} = \frac{1}{\sqrt{N}} \approx \frac{\theta}{2}$$

The number of steps required to find the desired file

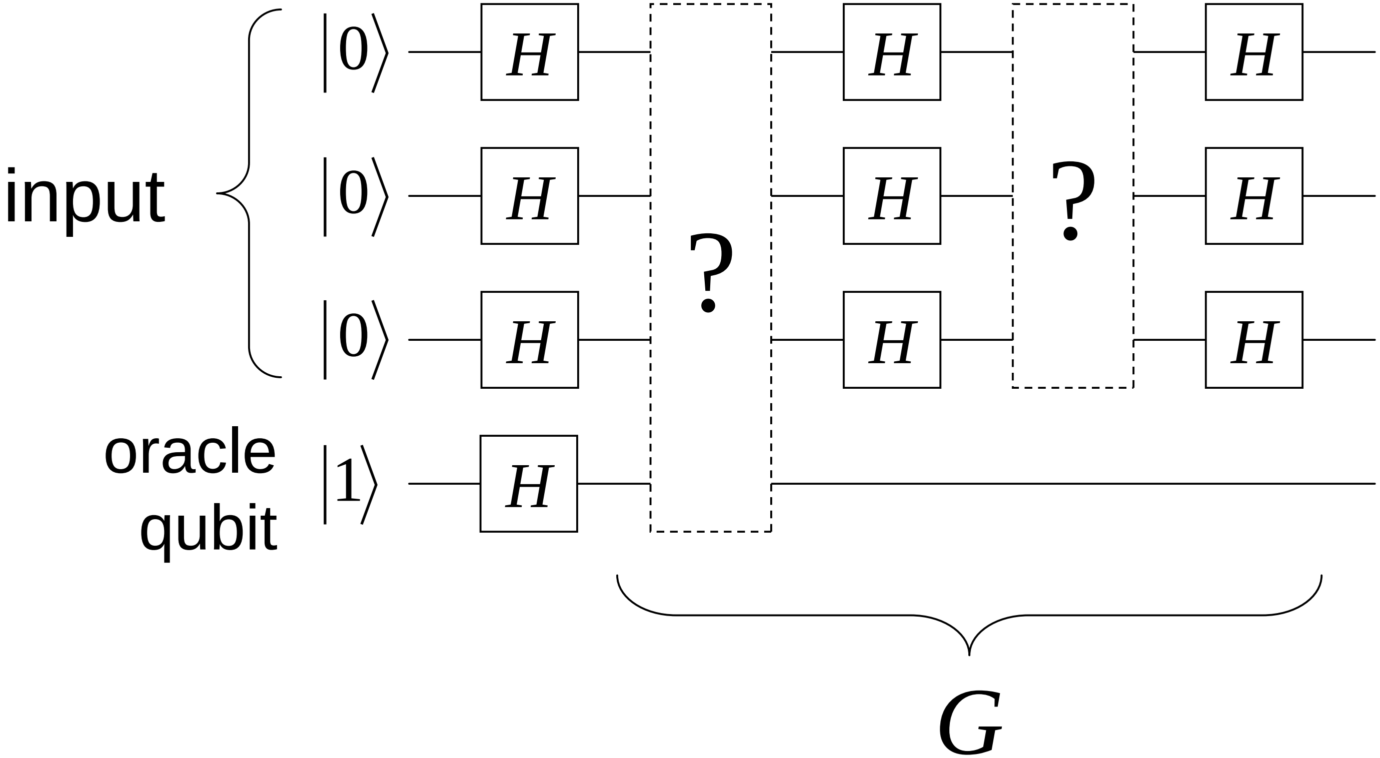
$$k' \approx \frac{\pi}{4}(\sqrt{N}-1) = O(\sqrt{N})$$

Optimality

- Classical algorithms take $O(N)$ operations for searching N items
- Grover's algorithm can search N items by calling the oracle only $O(N^{1/2})$ times
- It is shown that Grover's algorithm is optimal, i.e., any quantum algorithms require at least $O(N^{1/2})$ times oracle callings for searching
- The proof is beyond the scope of this introductory lecture

Quiz

Complete the quantum circuit for 3-bit Grover to search the file #7



Answer