Quantum Teleportation

School on Quantum Computing @Yagami

Day 1, Lesson 2

10:30-11:30, March 22, 2005

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Outline

- What is quantum teleportation?
  - State preparation
  - Bell measurement
  - Classical communication
  - Recovery operation
- Quantum circuit for QT
- SWAP revisited

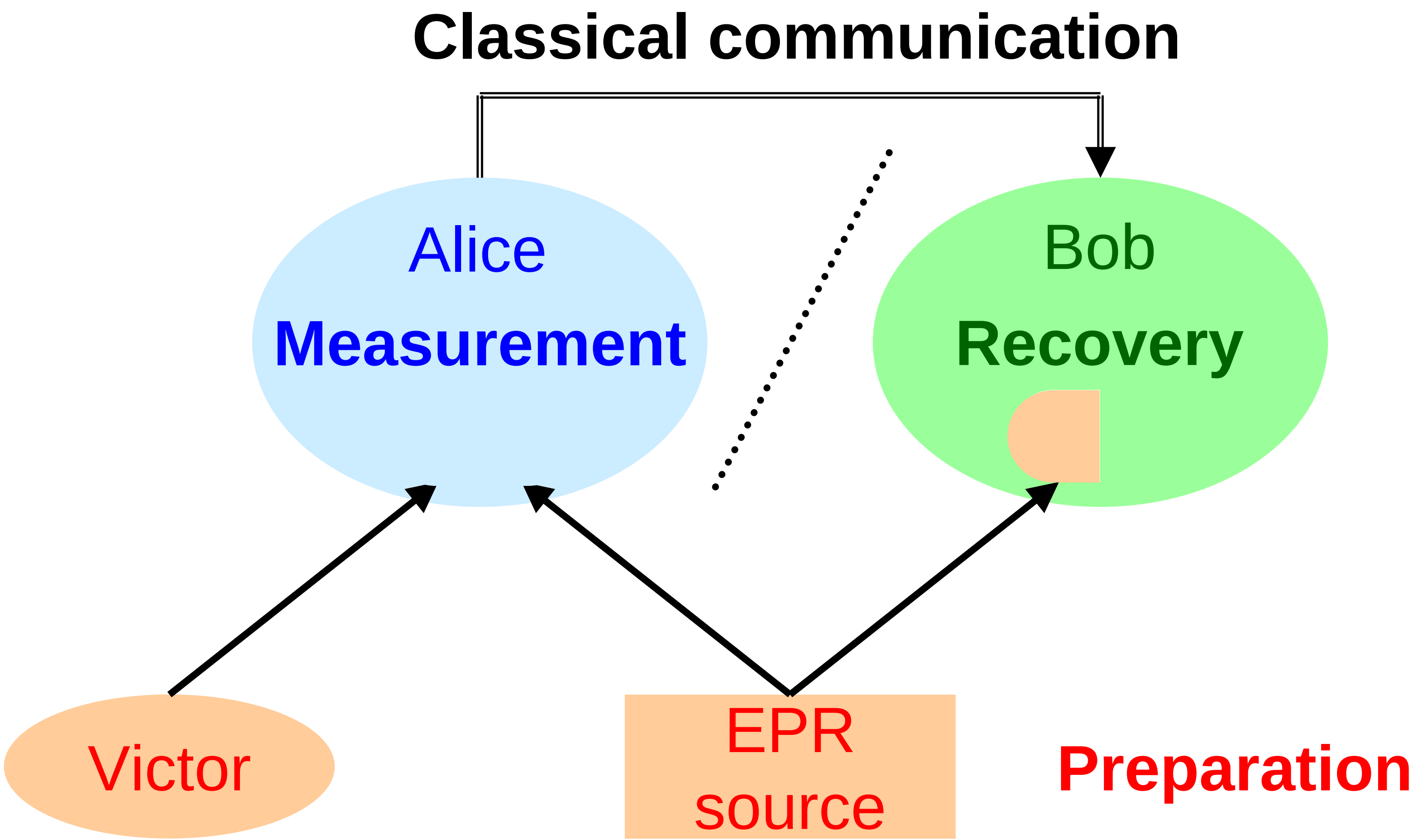
The inventors

Phys. Rev. Lett. **70** 1895 (1993)

Charles H. Bennett
Gilles Brassard
Claude Crépeau
Richard Jozsa
Asher Peres
William K. Wootters

© Aya Furuta

What is quantum teleportation?



$$|\psi\rangle_V = \alpha|0\rangle_V + \beta|1\rangle_V$$

$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

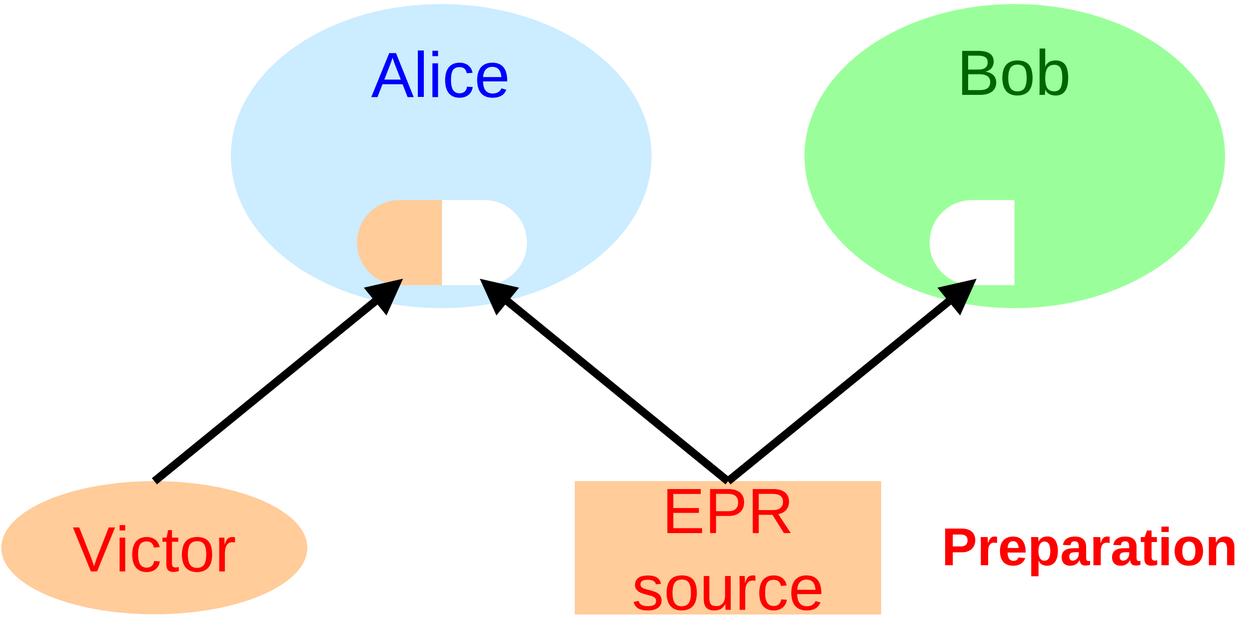
Step 1: State preparation

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B + \frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B + \frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B$$

- ✓ Alice and Bob share an entangled EPR pair in advance
- ✓ Alice mixes her state with Victor's unknown state $|\psi\rangle$, which she wants to deliver to Bob

$$|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$
$$|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$$
$$|\beta_{10}\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$
$$|\beta_{11}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$$
$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$$
$$XZ|\psi\rangle = \alpha|1\rangle - \beta|0\rangle$$



Check

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B + \frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B + \frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B$$

Expand the left-hand side

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = (\alpha|0\rangle + \beta|1\rangle)\otimes\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

Expand each term of the right-hand side

$$|\beta_{00}\rangle_{VA}|\psi\rangle_B = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\otimes(\alpha|0\rangle + \beta|1\rangle) = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\beta}{\sqrt{2}}|001\rangle + \frac{\alpha}{\sqrt{2}}|110\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$|\beta_{10}\rangle_{VA}Z|\psi\rangle_B = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\otimes(\alpha|0\rangle - \beta|1\rangle) = \frac{\alpha}{\sqrt{2}}|000\rangle - \frac{\beta}{\sqrt{2}}|001\rangle - \frac{\alpha}{\sqrt{2}}|110\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

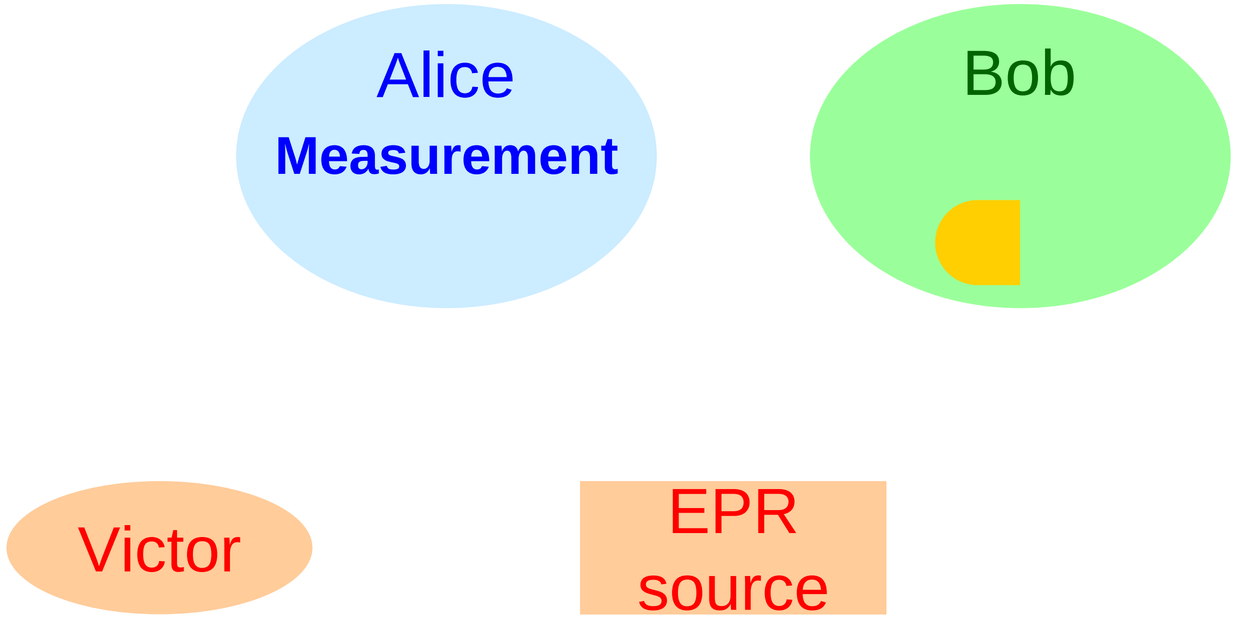
$$|\beta_{01}\rangle_{VA}X|\psi\rangle_B = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\otimes(\alpha|1\rangle + \beta|0\rangle) = \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|010\rangle + \frac{\alpha}{\sqrt{2}}|101\rangle + \frac{\beta}{\sqrt{2}}|100\rangle$$

$$|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\otimes(\alpha|1\rangle - \beta|0\rangle) = \frac{\alpha}{\sqrt{2}}|011\rangle - \frac{\beta}{\sqrt{2}}|010\rangle - \frac{\alpha}{\sqrt{2}}|101\rangle + \frac{\beta}{\sqrt{2}}|100\rangle$$

Step 2: Bell measurement by Alice

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B + \boxed{\frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B} + \frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B$$

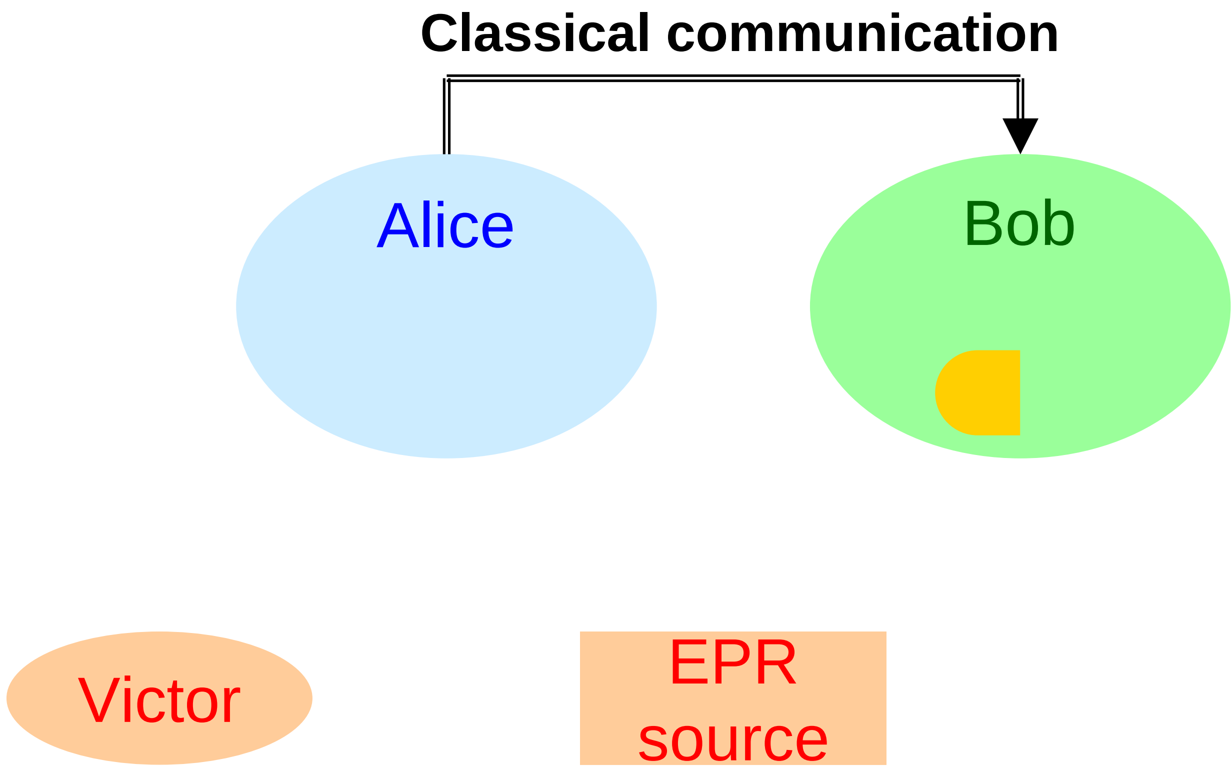
- ✓ Suppose Alice obtained the result $xy = 01$
- ✓ Bob’s state is now fixed as $X|\psi\rangle_B$, though he does not know about it



Step 3: Classical communication

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \cancel{\frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B} + \boxed{\frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B} + \cancel{\frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B} + \cancel{\frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B}$$

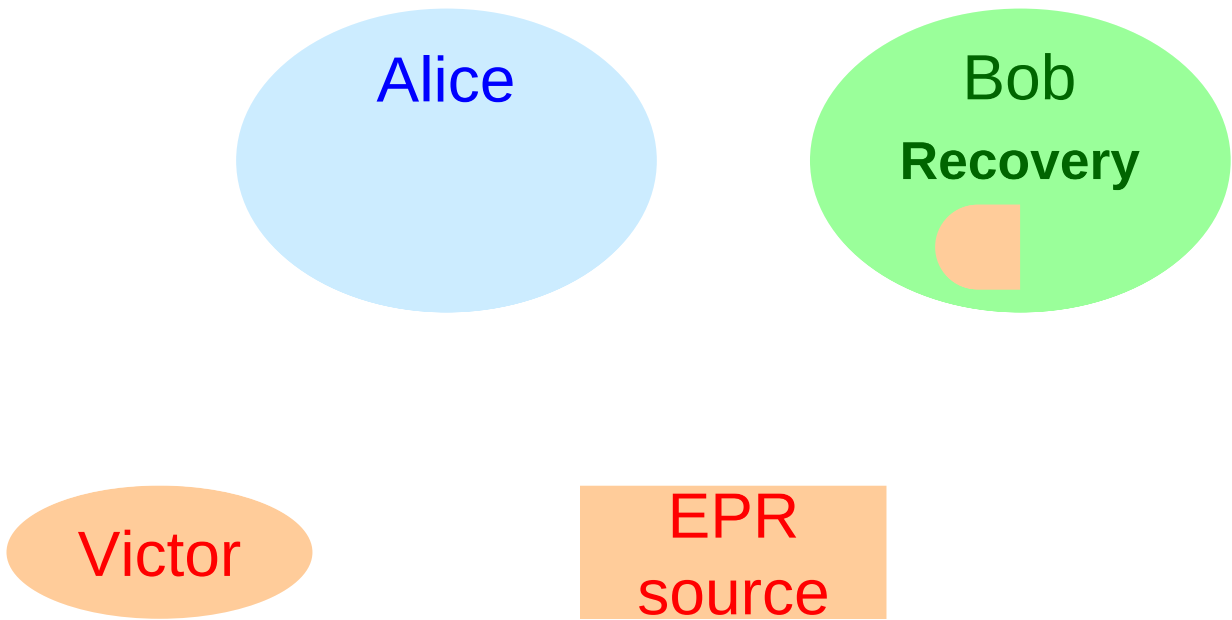
- ✓ Alice sends her classical result to Bob over a classical channel



Step 4: Recovery operation by Bob

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \cancel{\frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B} + \boxed{\frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B} + \cancel{\frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B} + \cancel{\frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B}$$

- ✓ Based on the information from Alice, Bob implements Pauli-X to his state
- ✓ Bob's state is now $X(X|\psi\rangle_B) = |\psi\rangle_B$, thus QT is completed

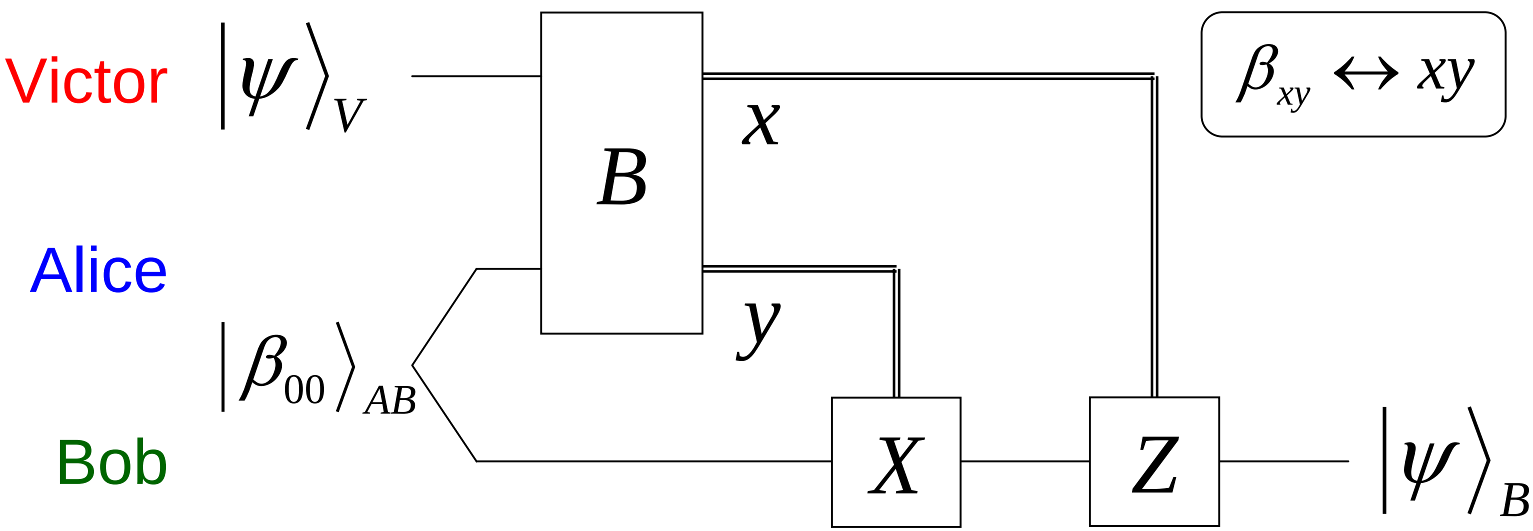


Quantum circuit for QT

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B + \frac{1}{2}|\beta_{01}\rangle_{VA} X|\psi\rangle_B + \frac{1}{2}|\beta_{10}\rangle_{VA} Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{VA} XZ|\psi\rangle_B$$

Step 2 Measurement

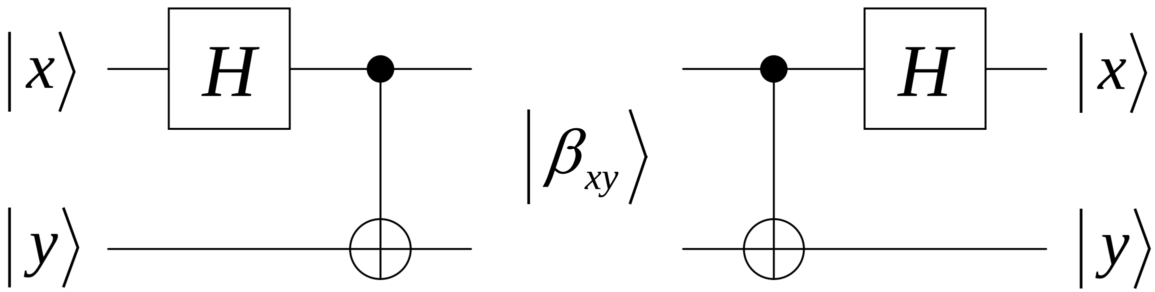
Step 3 Classical communication



Step 1 Preparation

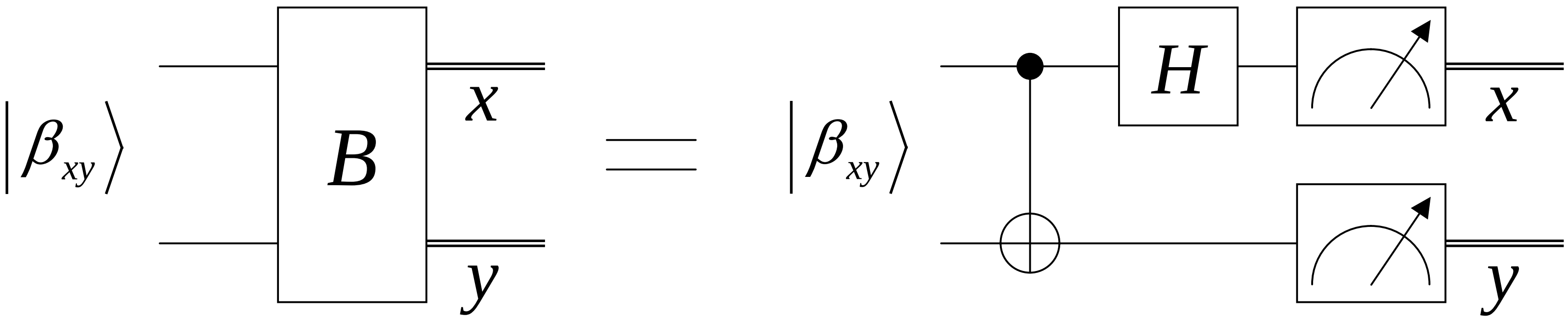
Step 4 Recovery

Base transformation



$$|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$
$$|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$$
$$|\beta_{10}\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$
$$|\beta_{11}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

$$|xy\rangle \overset{H_1}{\longleftrightarrow} \frac{|0y\rangle + (-1)^x |1y\rangle}{\sqrt{2}} \overset{C_{12}}{\longleftrightarrow} \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} = |\beta_{xy}\rangle$$



Measurement & recovery

$$|\psi\rangle_V|\beta_{00}\rangle_{AB} = \frac{1}{2}|\beta_{00}\rangle_{VA}|\psi\rangle_B + \frac{1}{2}|\beta_{01}\rangle_{VA}X|\psi\rangle_B + \frac{1}{2}|\beta_{10}\rangle_{VA}Z|\psi\rangle_B + \frac{1}{2}|\beta_{11}\rangle_{VA}XZ|\psi\rangle_B$$

$$= \frac{1}{2}\sum_{x,y}|\beta_{xy}\rangle_{VA}X^yZ^x|\psi\rangle_B$$

Base transformation

$$\frac{1}{2}\sum_{x,y}|xy\rangle_{VA}X^yZ^x|\psi\rangle_B \xrightarrow{Z_B^x X_B^y} \frac{1}{2}\sum_{x,y}|xy\rangle_{VA}|\psi\rangle_B$$

Recover

Measure

Measure

$$x', y', X^{y'}Z^{x'}|\psi\rangle_B \xrightarrow{Z_B^{x'} X_B^{y'}} x', y', |\psi\rangle_B$$

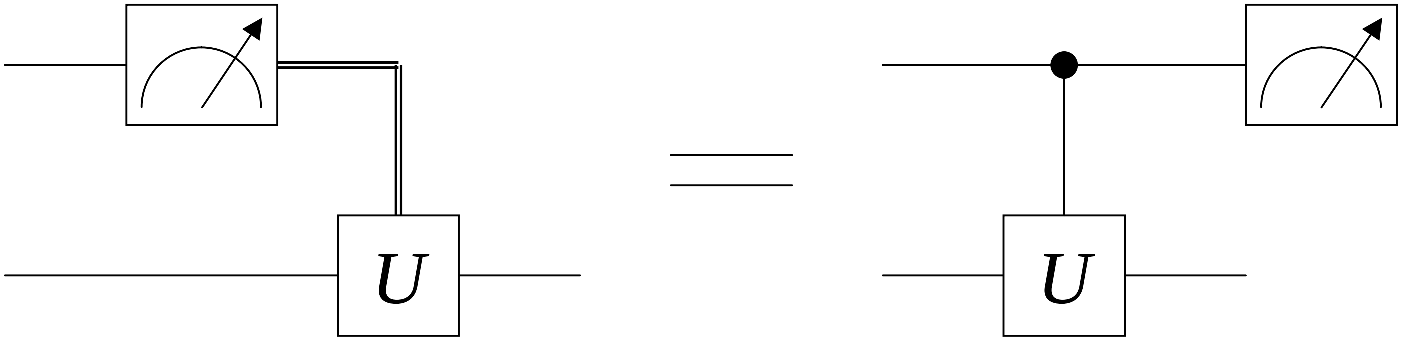
Recover

Whether we first measure or recover does not affect the result

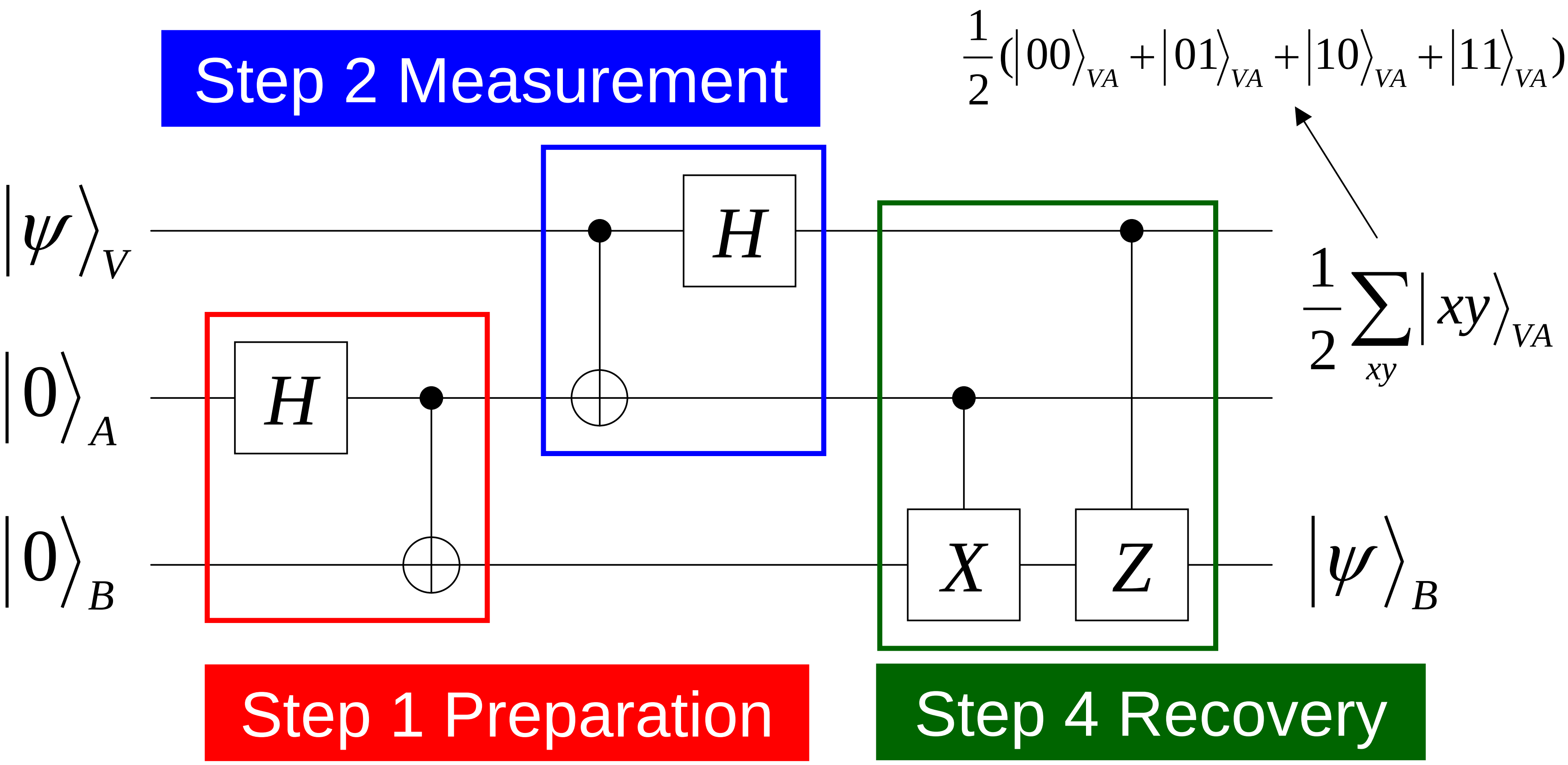
Principle of deferred measurement

Measurement commutes with controls

If the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations



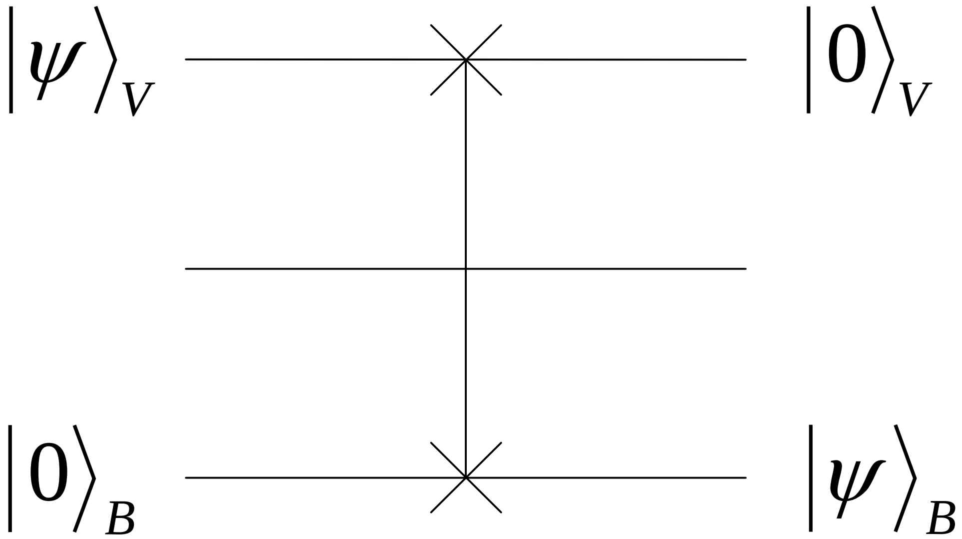
Quantum circuit for QT



Now this circuit may not be interpreted as “teleportation”, but is equivalent to the previous QT circuit

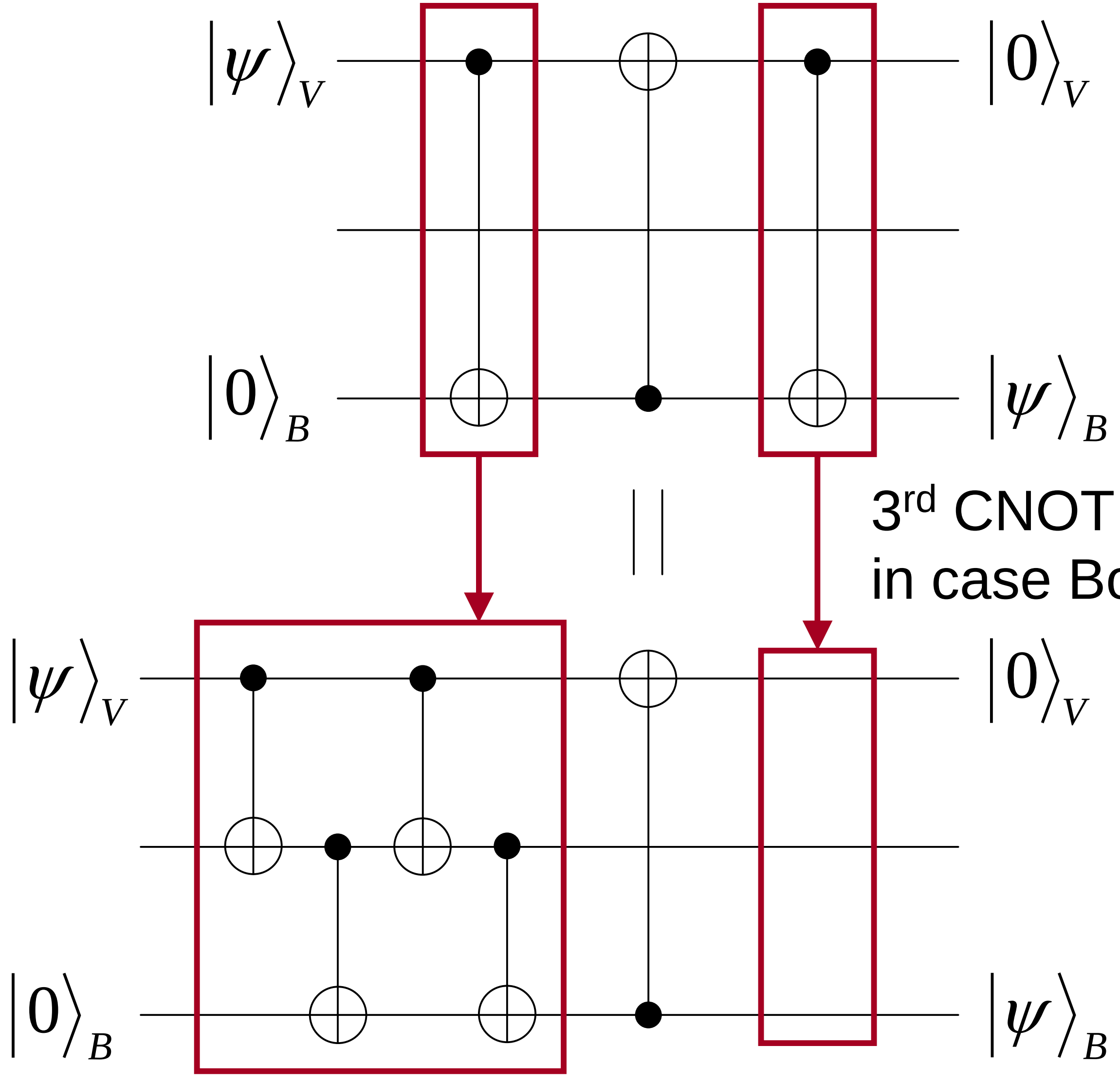
SWAP revisited

Is the SWAP another quantum teleportation?



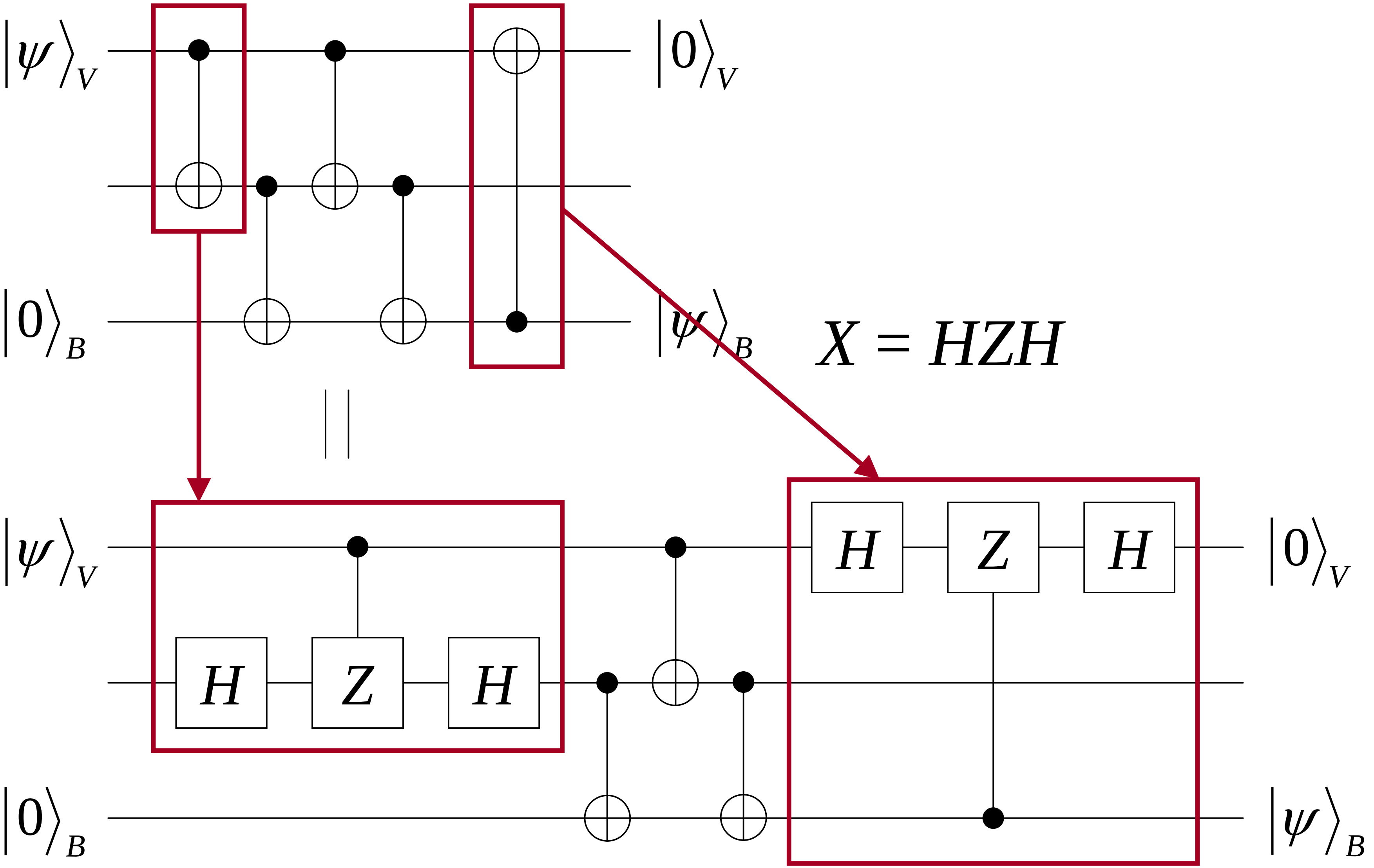
- ✓ No classical communication
- ✓ Alice is not involved in the process

From SWAP to QT

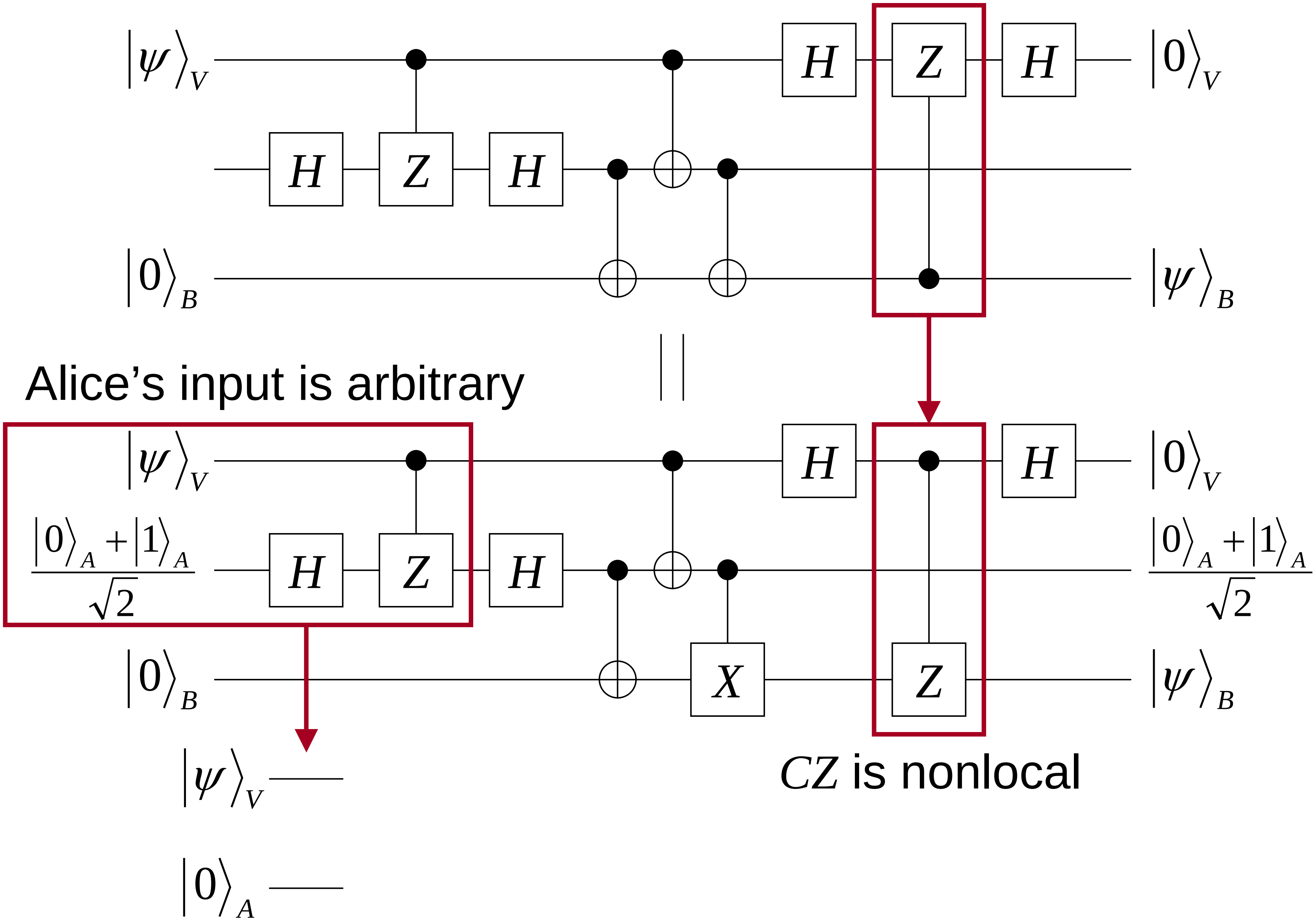


3rd CNOT is unnecessary in case Bob's input is 0

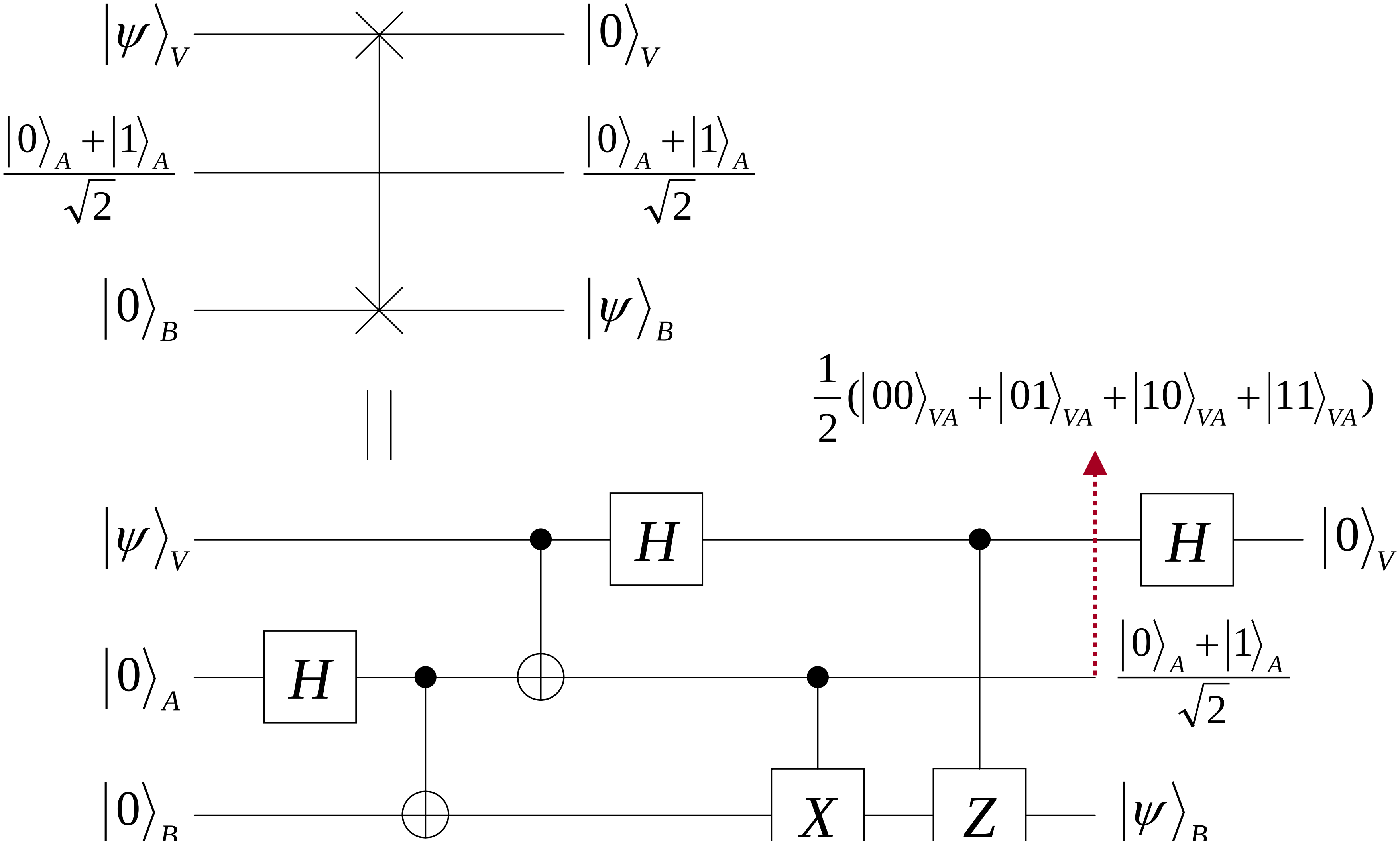
From SWAP to QT



From SWAP to QT



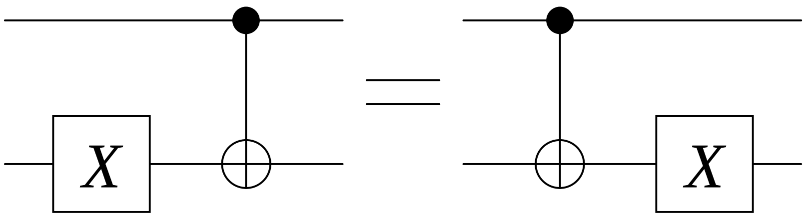
From SWAP to QT



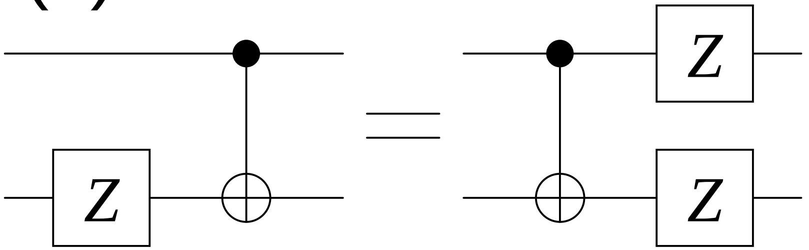
Quiz 1

Prove the following circuit identities

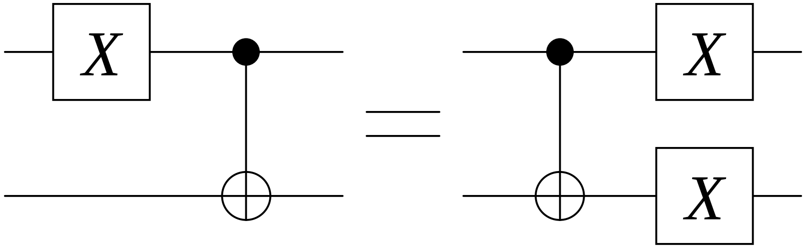
(1)



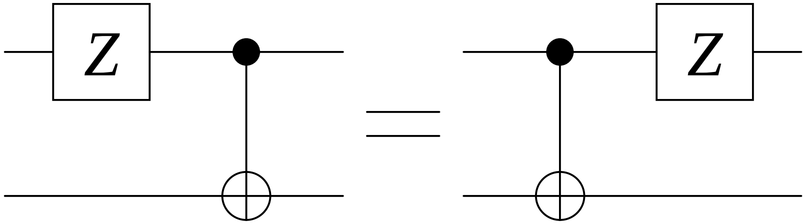
(2)



(3)

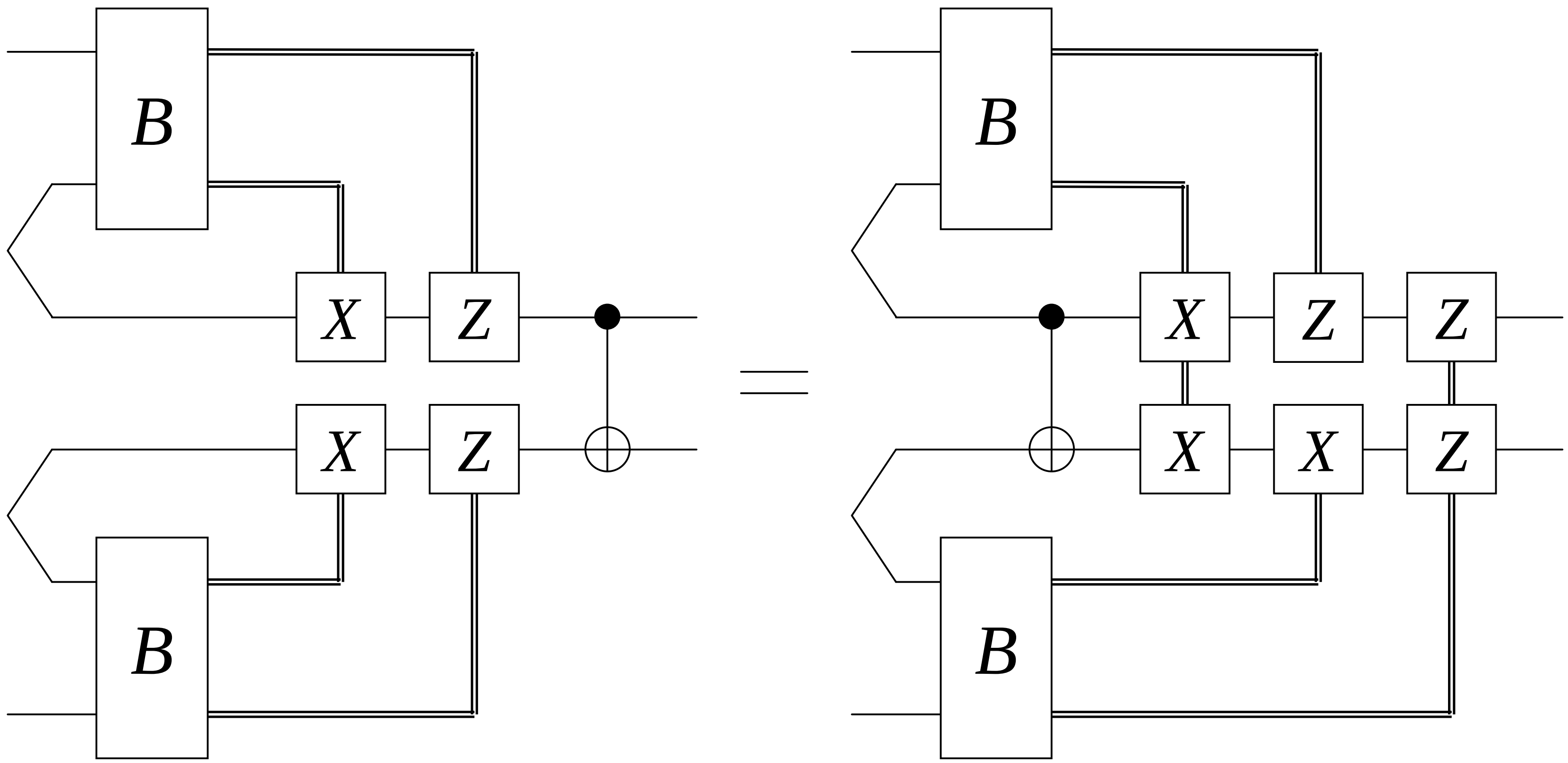


(4)



Quiz 2

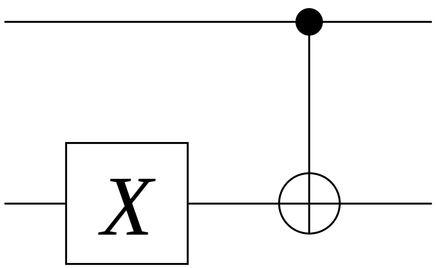
Prove the following circuit identity



D. Gottesman & I. L. Chuang, Nature **402** 390 (1999)

Answer

(1)



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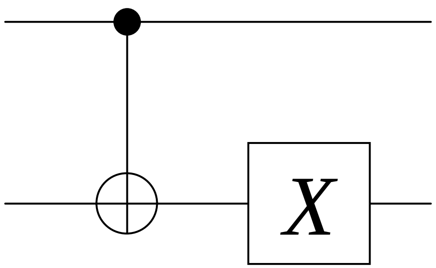


$$C_{12}X_2|a\rangle|b\rangle = C_{12}|a\rangle|b\oplus 1\rangle$$
$$= |a\rangle|a\oplus b\oplus 1\rangle$$

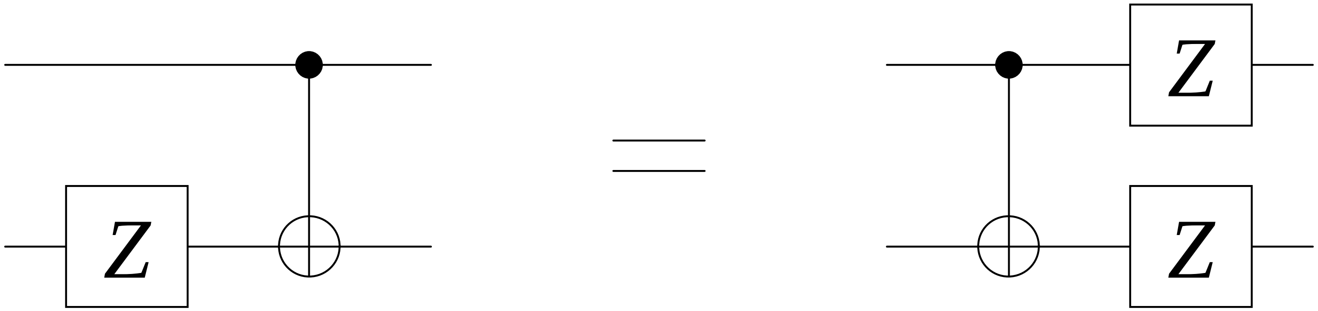
$$X_2C_{12}|a\rangle|b\rangle = X_2|a\rangle|a\oplus b\rangle$$
$$= |a\rangle|a\oplus b\oplus 1\rangle$$

Two Pauli-X gates commute

FYI… $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer

(2)



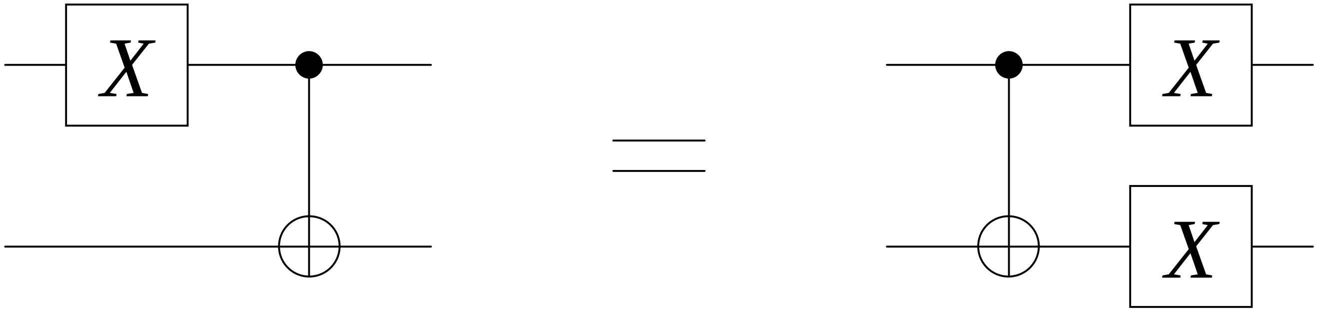
$$C_{12}Z_2|a\rangle|b\rangle = (-1)^b C_{12}|a\rangle|b\rangle = (-1)^b|a\rangle|a\oplus b\rangle$$

$$Z_1Z_2C_{12}|a\rangle|b\rangle = Z_1Z_2|a\rangle|a\oplus b\rangle = (-1)^{a+a+b}|a\rangle|a\oplus b\rangle = (-1)^b|a\rangle|a\oplus b\rangle$$

Pauli-Z and Pauli-X do not commute, and the minus sign need be compensated

Answer

(3)



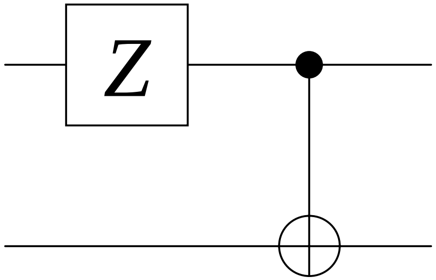
$$C_{12}X_1|a\rangle|b\rangle = C_{12}|a\oplus 1\rangle|b\rangle$$
$$= |a\oplus 1\rangle|a\oplus b\oplus 1\rangle$$

$$X_1X_2C_{12}|a\rangle|b\rangle = X_1X_2|a\rangle|a\oplus b\rangle$$
$$= |a\oplus 1\rangle|a\oplus b\oplus 1\rangle$$

Flip of the control bit need be compensated by the additional NOT on the target bit

Answer

(4)



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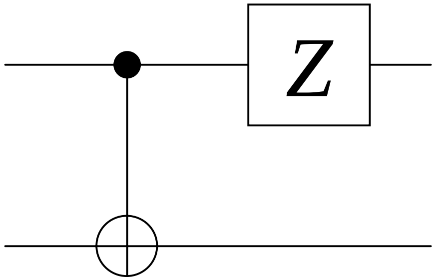

$$C_{12}Z_1|a\rangle|b\rangle = (-1)^a C_{12}|a\rangle|b\rangle$$
$$= (-1)^a |a\rangle|a \oplus b\rangle$$

$$Z_1 C_{12}|a\rangle|b\rangle = Z_1|a\rangle|a \oplus b\rangle$$
$$= (-1)^a |a\rangle|a \oplus b\rangle$$

Phase of the control bit does not matter

Answer