Qubit and Quantum Gates

School on Quantum Computing @Yagami

Day 1, Lesson 1

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Eisuke Abe

Department of Applied Physics and Physico-Informatics,

and CREST-JST, Keio University

From classical to quantum

Information is physical

- Rolf Landauer

- QUANTUM information or quantum INFORMATION?
- It depends on your background (physics or information science)
- Ultimately, you need both
- At the beginning, it would be better to keep one perspective (physics here)

References

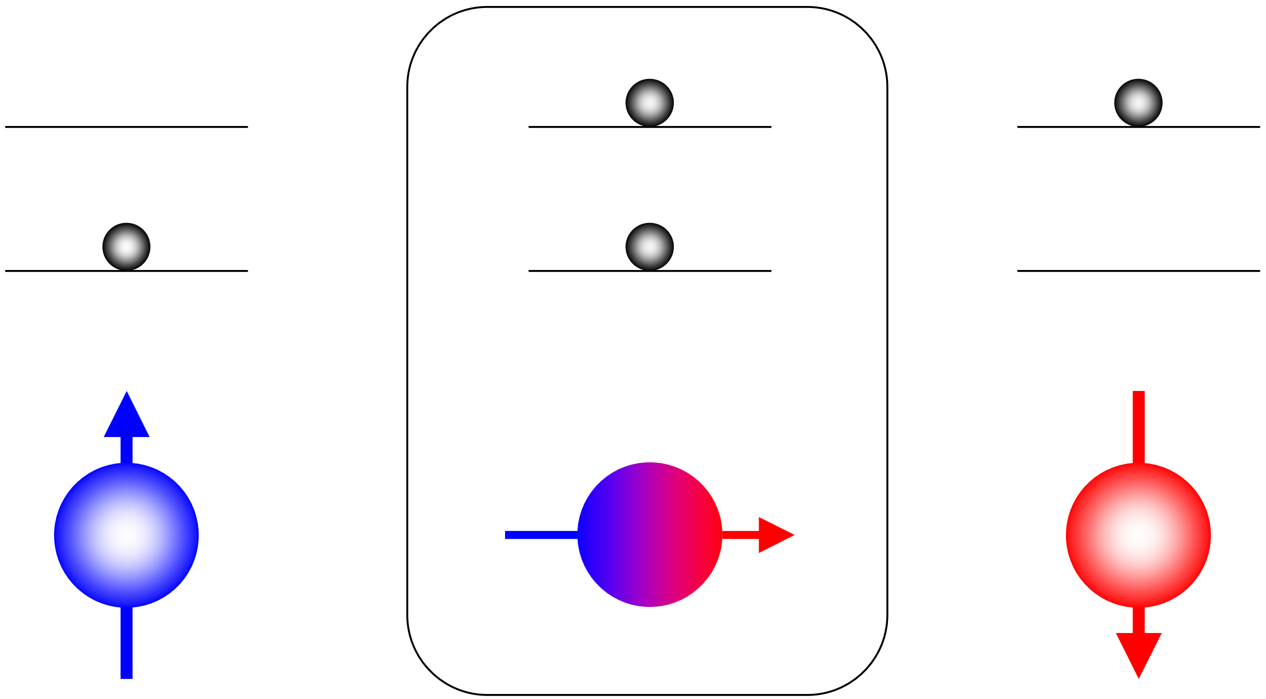
- Quantum Computation and Quantum Information (and references therein), M. A. Nielsen and I. L. Chuang, Cambridge University Press (2000)
  - Day 1, Lesson 1- Day 2, Lesson 2
- Physical Review A 65, 012320 (2001), N. D. Mermin
  - Day 1, Lesson 2
- L. M. K. Vandersypen, Ph. D Thesis (available at arXiv: quant-ph/0205193)
  - Day 2, Lesson 2

Outline

- Rules of the game
  - Quantum bit (State space)
  - Quantum gate (Unitary evolution)
    - NOT (X), Y, Z, Hadamard (H)
  - Measurement
  - Multiple-qubit (Tensor product)
    - CNOT, SWAP, Controlled-Z, Toffoli

Quantum bit

For physicists, "*quantum bit* (*qubit*)" is a synonym for "*quantum mechanical two-level system*"



$|g\rangle \equiv |0\rangle$ Superposition $|e\rangle \equiv |1\rangle$

Quantum bit

Vector notation for *computational basis* states

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

POSTULATE State space (Hilbert space)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$\alpha, \beta \in \mathbf{C}$: Probability amplitude

$|\alpha|^2 + |\beta|^2 = 1$: Probabilities sum to 1

Unitary evolution

POSTULATE

The evolution of a qubit system is described by a *unitary transformation* such as

$$\left|\psi(t_2)\right\rangle = U_{12}\left|\psi(t_1)\right\rangle$$

Hermitian conjugate: $A^\dagger = (A^T)^*$

Hermitian (self-adjoint): $A = A^\dagger$

Unitary: $UU^\dagger = I$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$$

Unitary evolution

Connection with the *Schrödinger equation*

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle \quad \Rightarrow \quad |\psi(t_2)\rangle = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]|\psi(t_1)\rangle \equiv U_{12}|\psi(t_1)\rangle$$

H: Hamiltonian of the qubit system (Hermitian)

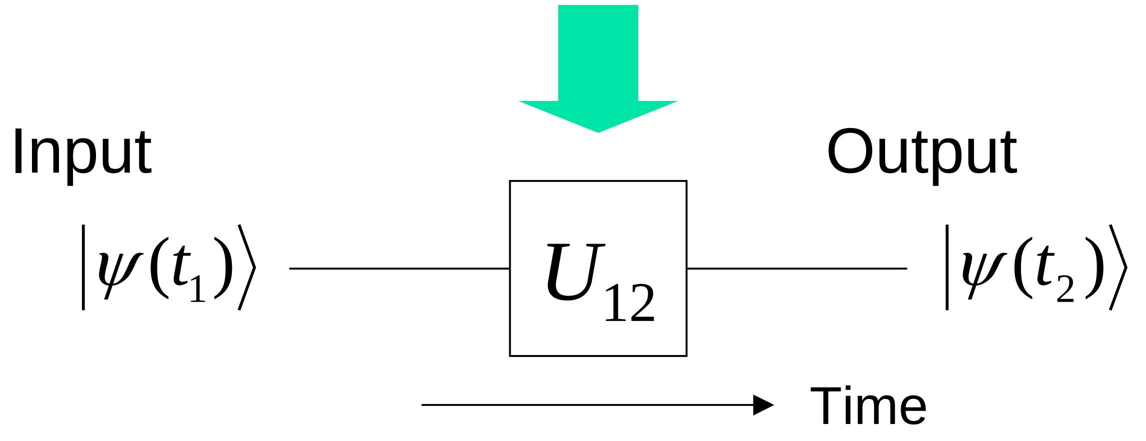
Exponential operator = unitary

Any unitary operator U can be realized in the form $U = \exp(iH)$ where H is some Hermitian operator

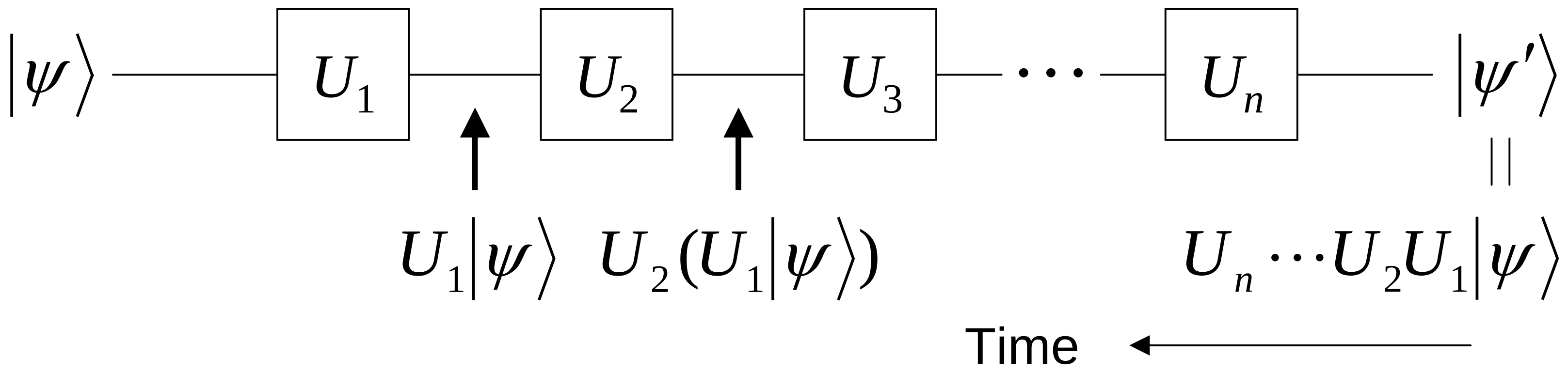
For now, actual physical systems that realize necessary Hamiltonians are NOT our interest

Quantum gate

$$|\psi(t_2)\rangle = U_{12}|\psi(t_1)\rangle$$



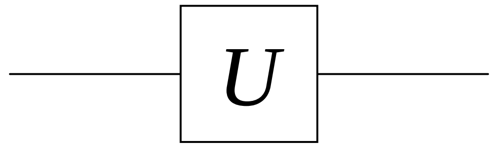
Successive implementation



Quantum gate

Input

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



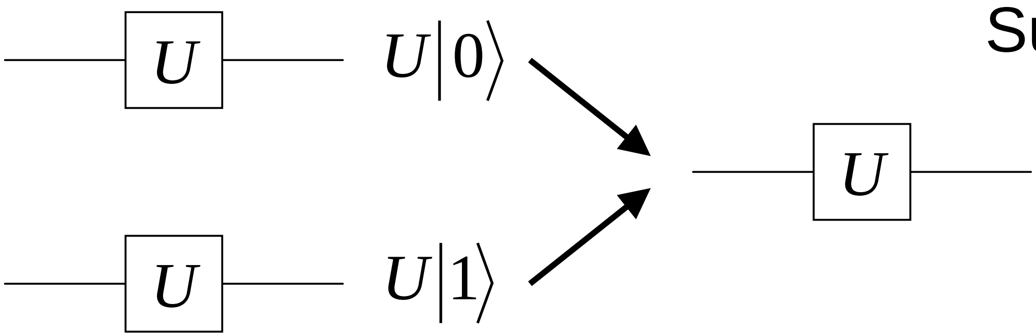
Output

$$U|\psi\rangle = U(\alpha|0\rangle + \beta|1\rangle)$$

We have *infinite* inputs, but it suffices to consider only the computational basis states

$|0\rangle$ → $U|0\rangle$

$|1\rangle$ → $U|1\rangle$



Superposition principle

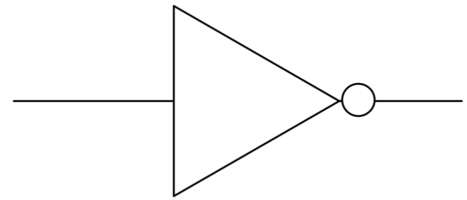
$$\alpha(U|0\rangle) + \beta(U|1\rangle) = U|\psi\rangle$$

NOT gate

Classical NOT

Input	output
0	1
1	0

a



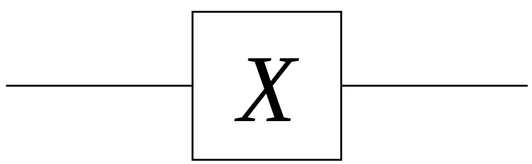
$$\bar{a} \equiv a \oplus 1$$

The only non-trivial one-bit gate in the classical case

$$\bar{0} = 0 \oplus 1 = 1$$
$$\bar{1} = 1 \oplus 1 = 0$$

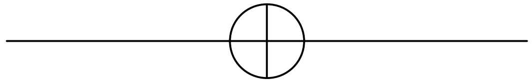
Quantum NOT

$|a\rangle$



$$X|a\rangle = |\bar{a}\rangle$$

or



Matrix representation

$$X|0\rangle = |\bar{0}\rangle = |1\rangle$$
$$X|1\rangle = |\bar{1}\rangle = |0\rangle$$
$$\Leftrightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrix representation

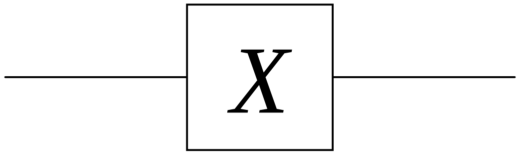
$$\begin{cases} X|0\rangle = |\overline{0}\rangle = |1\rangle \\ X|1\rangle = |\overline{1}\rangle = |0\rangle \end{cases} \Leftrightarrow \begin{cases} X\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ X\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \Leftrightarrow X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first column represents the final state of $|0\rangle$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

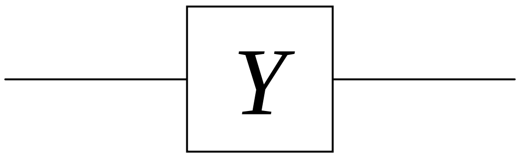
The second column represents the final state of $|1\rangle$

Pauli-X, Y, Z gates

$|a\rangle$ 

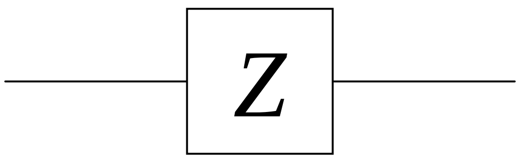
$|\bar{a}\rangle$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$|a\rangle$ 

$(-1)^a i|\bar{a}\rangle$

$$\begin{matrix} Y|0\rangle = i|1\rangle \\ Y|1\rangle = -i|0\rangle \end{matrix} \Leftrightarrow Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$|a\rangle$ 

$(-1)^a|a\rangle$

$$\begin{matrix} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{matrix} \Leftrightarrow Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hermitian

$$X^2 = Y^2 = Z^2 = I$$

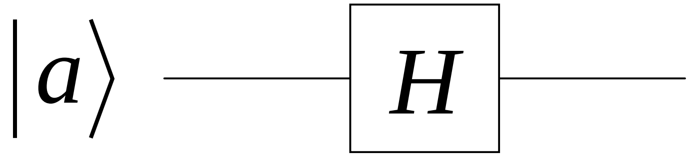
Commutation relations

$$[X, Y] = XY - YX = 2iZ \qquad \{X, Y\} = 0$$

$$[Y, Z] = 2iX \qquad \{Y, Z\} = 0$$

$$[Z, X] = 2iY \qquad \{Z, X\} = ZX + XZ = 0$$

Hadamard gate



$$\frac{1}{\sqrt{2}}\sum_{b=0,1}(-1)^{a\cdot b}|b\rangle = \frac{|0\rangle + (-1)^{a}|1\rangle}{\sqrt{2}}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\sum_{b=0,1}|b\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\sum_{b=0,1}(-1)^{b}|b\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\Leftrightarrow \quad H\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \; H\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\Leftrightarrow \quad H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

Hermitian

$$H^{2} = I$$

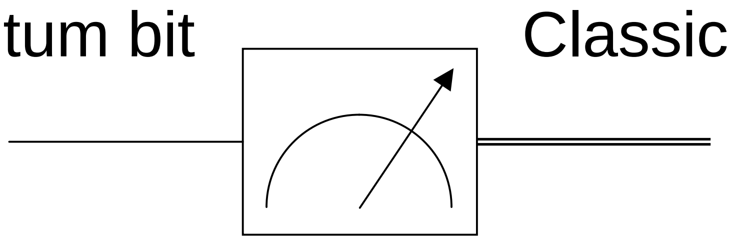
Circuit identities

$$HXH = Z \quad HYH = -Y \quad HZH = X$$

Measurement gate

POSTULATE

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



a

0 with probability $|\alpha|^2$, or
1 with probability $|\beta|^2$

Mathematical description

- ✓ General measurement
- ✓ Projective measurement
- ✓ POVM

Multiple-qubit

How do we describe multiple-qubit states?

Speculation…

- ✓ Computational basis states for two-qubit states may be written as $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
- ✓ We require them to be orthogonal, so they may be written as

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

POSTULATE

A multiple-qubit state is the *tensor product* of the component qubit systems

Tensor product

Matrix representation

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ a_2 \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$$

Computational basis set for 2-qubit states

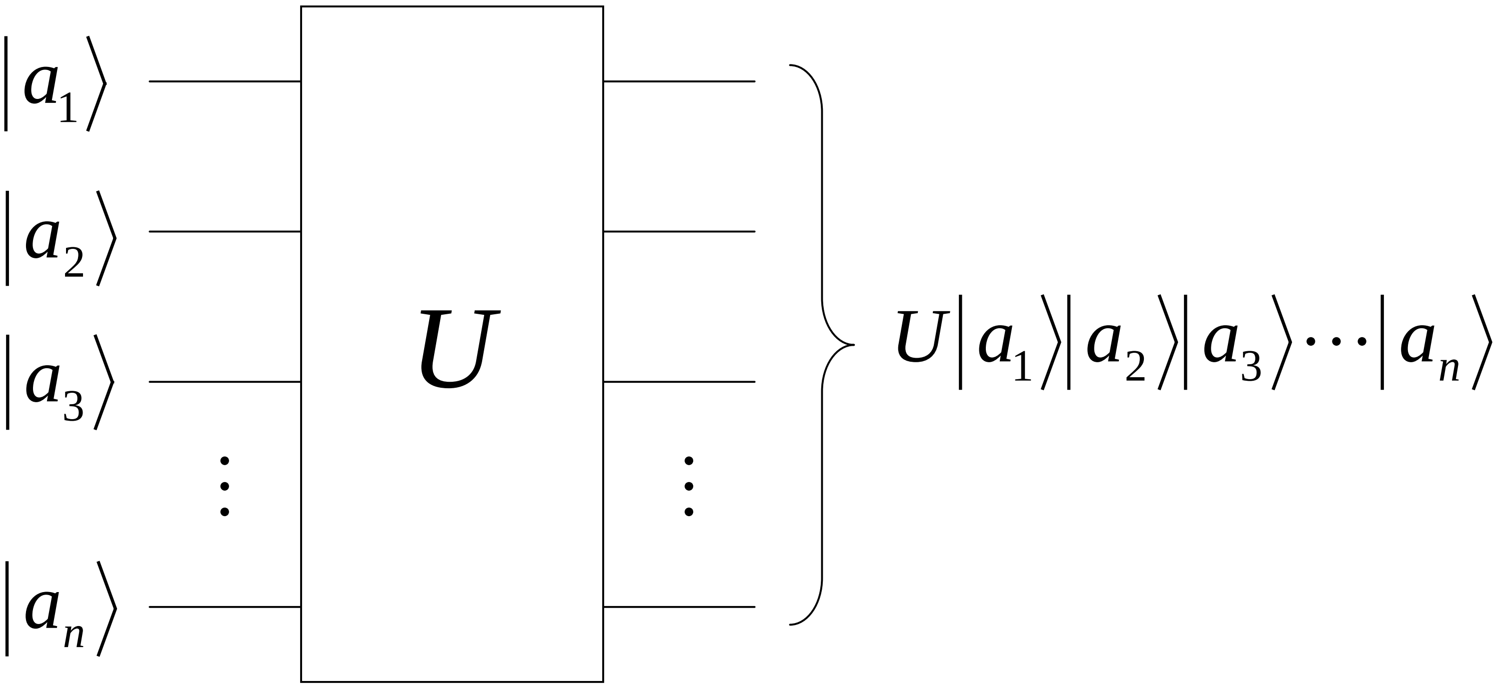
$$|00\rangle = |0\rangle|0\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = |1\rangle|0\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle|1\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|11\rangle = |1\rangle|1\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

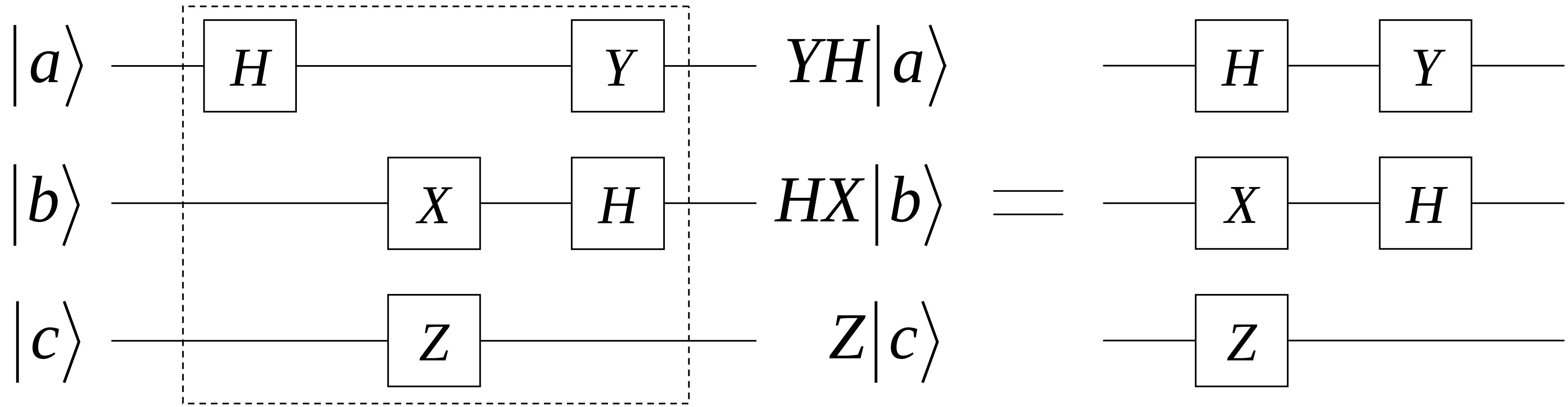
Multiple-qubit gates



$|a_1 a_2 \ldots a_n\rangle$: 2^n-dimensional vector

U : 2^n by 2^n unitary matrix

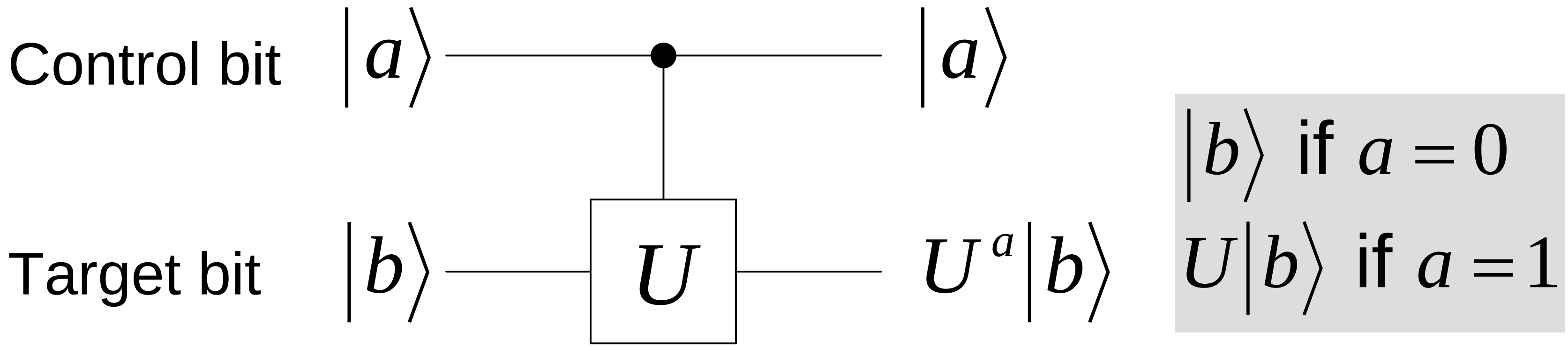
Independent gates



$$|a\rangle|b\rangle|c\rangle \rightarrow \underbrace{(YH \otimes HX \otimes Z)}_{U \text{ (8 by 8 unitary matrix)}}|a\rangle|b\rangle|c\rangle = (YH|a\rangle) \otimes (HX|b\rangle) \otimes Z|c\rangle$$

$$A \otimes B = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 \times B & a_3 \times B \\ a_2 \times B & a_4 \times B \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_3 & a_3 b_1 & a_3 b_3 \\ a_1 b_2 & a_1 b_4 & a_3 b_2 & a_3 b_4 \\ a_2 b_1 & a_2 b_3 & a_4 b_1 & a_4 b_3 \\ a_2 b_2 & a_2 b_4 & a_4 b_2 & a_4 b_4 \end{bmatrix}$$

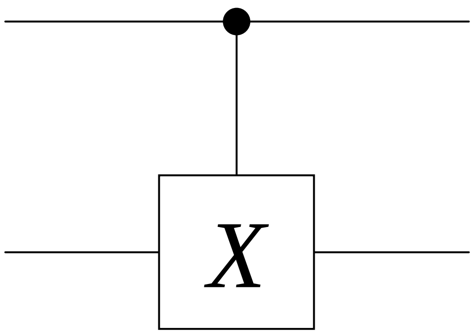
Controlled-U gates



- ✓ U can be an arbitrary single-qubit gate
- ✓ U works only when $a = 1$
- ✓ $U^a|b\rangle$ is just a formal expression

CNOT gate

$|a\rangle$ → $|a\rangle$

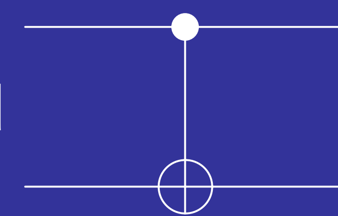


$|b\rangle$ → $X^{a}|b\rangle = |b \oplus a\rangle$

$|b\rangle = |b \oplus 0\rangle$ if $a = 0$

$|\bar{b}\rangle = |b \oplus 1\rangle$ if $a = 1$

Frequently used



$$C_{12}|00\rangle = |0\rangle|0 \oplus 0\rangle = |00\rangle$$

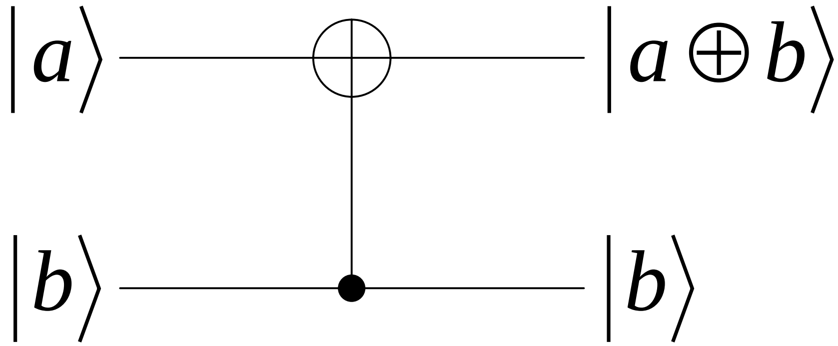
$$C_{12}|01\rangle = |0\rangle|1 \oplus 0\rangle = |01\rangle$$

$$C_{12}|10\rangle = |1\rangle|0 \oplus 1\rangle = |11\rangle$$

$$C_{12}|11\rangle = |1\rangle|1 \oplus 1\rangle = |10\rangle$$

$$\Leftrightarrow C_{12}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad C_{12}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad C_{12}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \quad C_{12}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \Leftrightarrow C_{12} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

CNOT gate



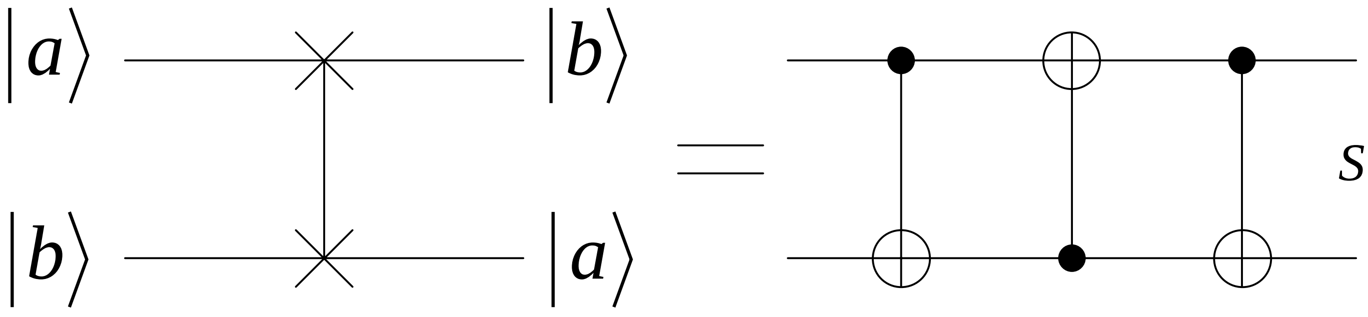
$$C_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} C_{21}|00\rangle &= |0 \oplus 0\rangle|0\rangle = |00\rangle \\ C_{21}|01\rangle &= |0 \oplus 1\rangle|1\rangle = |11\rangle \\ C_{21}|10\rangle &= |1 \oplus 0\rangle|0\rangle = |10\rangle \\ C_{21}|11\rangle &= |1 \oplus 1\rangle|1\rangle = |01\rangle \end{aligned} \Leftrightarrow C_{21}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, C_{21}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, C_{21}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, C_{21}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Never mistake…

$$C_{21} \neq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SWAP gate



$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|a\rangle|b\rangle \xrightarrow{C_{12}} |a\rangle|b\oplus a\rangle \qquad a\oplus a = 0$$ 1

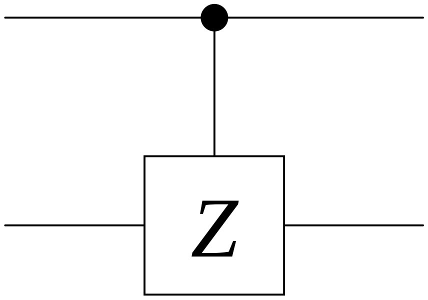
$$\xrightarrow{C_{21}} |a\oplus(b\oplus a)\rangle|b\oplus a\rangle = |b\rangle|b\oplus a\rangle$$ 2 3

$$\xrightarrow{C_{12}} |b\rangle|(b\oplus a)\oplus b\rangle = |b\rangle|a\rangle$$ 4

To implement SWAP, we need to…

1. Encode information on $|a\rangle$ into 2nd qubit
2. Erase information on $|a\rangle$ from 1st qubit
3. Encode information on $|b\rangle$ into 1st qubit
4. Erase information on $|b\rangle$ from 2nd qubit

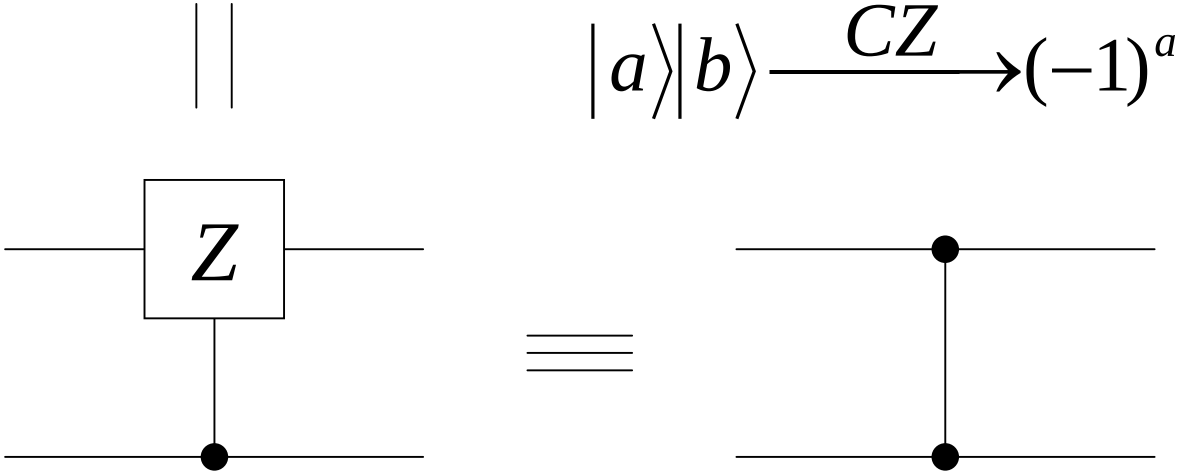
Controlled-Z gate

$|a\rangle$  $|a\rangle$

$|b\rangle$ $(-1)^{a\cdot b}|b\rangle$

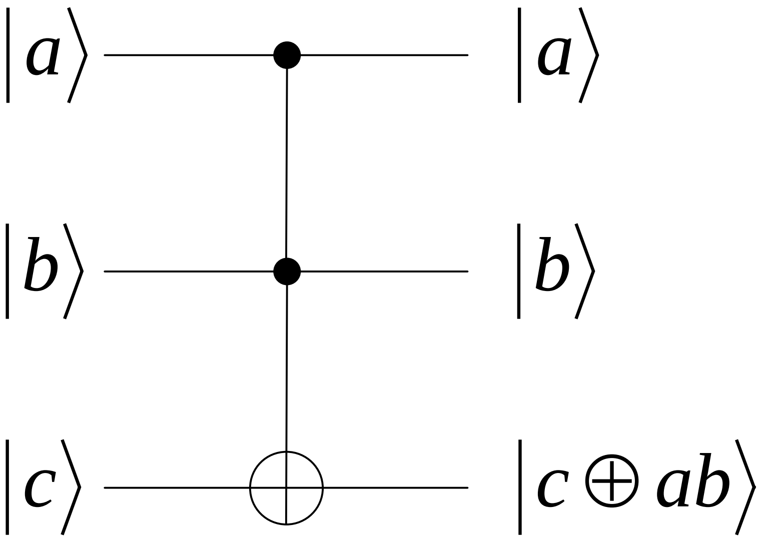
$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$|a\rangle|b\rangle \xrightarrow{CZ} (-1)^{a\cdot b}|a\rangle|b\rangle$$



Controlled-Z is *nonlocal*

Toffoli



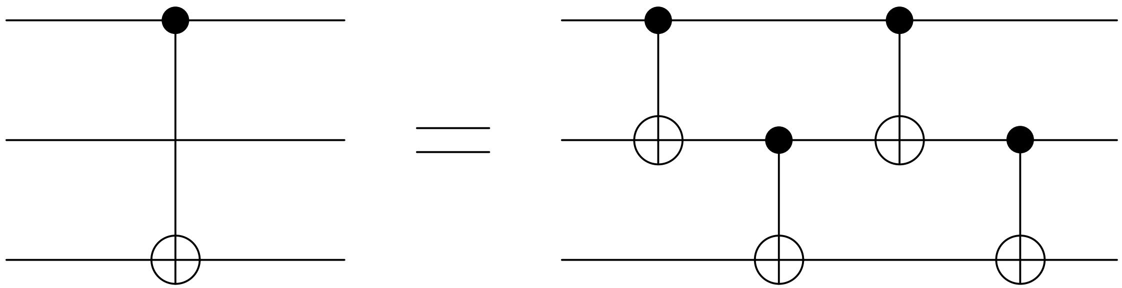
$$Toffoli = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$ab = 1 \Leftrightarrow a = 1 \wedge b = 1$$

Toffoli is often referred to as "controlled-controlled-NOT (C^2-NOT)"

Quiz

Prove the following circuit identity



Also prove the followings

$$H^2 = I$$

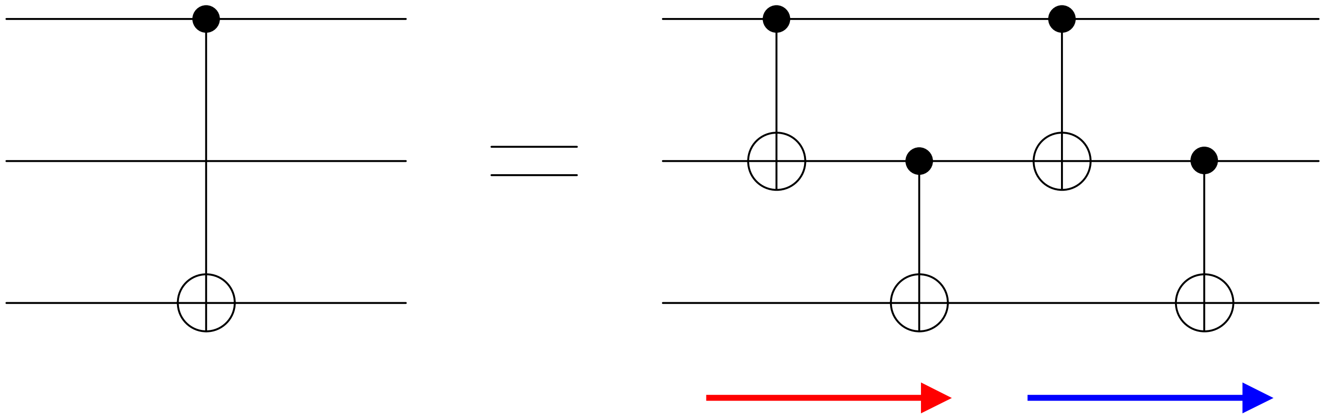
$$HZH = X$$

Use the following expressions for quantum gates

$$C_{12}|a\rangle|b\rangle = |a\rangle|b \oplus a\rangle$$

$$H|a\rangle = \frac{1}{\sqrt{2}}\sum_b (-1)^{a\cdot b}|b\rangle, \quad Z|a\rangle = (-1)^a|a\rangle$$

Answer



$$|a\rangle|b\rangle|c\rangle \xrightarrow{C_{12}} |a\rangle|b\oplus a\rangle|c\rangle$$

$$\xrightarrow{C_{23}} |a\rangle|b\oplus a\rangle|c\oplus b\oplus a\rangle$$

$$\xrightarrow{C_{12}} |a\rangle|(b\oplus a)\oplus a\rangle|c\oplus b\oplus a\rangle = |a\rangle|b\rangle|c\oplus b\oplus a\rangle$$

$$\xrightarrow{C_{23}} |a\rangle|b\rangle|(c\oplus b\oplus a)\oplus b\rangle = |a\rangle|b\rangle|c\oplus a\rangle$$

Cascade implementation

Cascade erasure

Answer

$$HH|a\rangle = H\left(\frac{1}{\sqrt{2}}\sum_b (-1)^{a\cdot b}|b\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\sum_b (-1)^{a\cdot b}\left(\frac{1}{\sqrt{2}}\sum_c (-1)^{b\cdot c}|c\rangle\right) = \frac{1}{2}\sum_b\sum_c (-1)^{(a+c)\cdot b}|c\rangle$$

$$= \frac{1}{2}\sum_b \left(|a\rangle + (-1)^b|\overline{a}\rangle\right)$$

$$(a+c)\cdot b = \begin{cases} 0 & (c = a) \\ b & (c = \overline{a}) \end{cases}$$

$$= \frac{1}{2}\left(|a\rangle + |\overline{a}\rangle + |a\rangle - |\overline{a}\rangle\right) = |a\rangle$$

Constructive and destructive interferences

Answer

$$HZH|a\rangle = HZ\left(\frac{1}{\sqrt{2}}\sum_b (-1)^{a\cdot b}|b\rangle\right)$$

$$= H\left(\frac{1}{\sqrt{2}}\sum_b (-1)^{a\cdot b+b}|b\rangle\right)$$

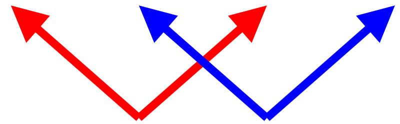
$$a\cdot b + b = (a+1)\cdot b = \overline{a}\cdot b$$

$$= \frac{1}{2}\sum_b\sum_c (-1)^{(\overline{a}+c)\cdot b}|c\rangle$$

$$(\overline{a}+c)\cdot b = \begin{cases} 0 & (c=\overline{a}) \\ b & (c=a) \end{cases}$$

$$= \frac{1}{2}\sum_b (|\overline{a}\rangle + (-1)^b|a\rangle)$$

$$= \frac{1}{2}\left(|\overline{a}\rangle + |a\rangle + |\overline{a}\rangle - |a\rangle\right) = |\overline{a}\rangle$$



Constructive and destructive interferences