

量子情報基礎

阿部 英介

理化学研究所 創発物性科学研究センター

応用物理情報特別講義A

2019年度春学期後半 金曜4限@14-202

参考書

- J. J. Sakurai (1993)
 - ***“Modern Quantum Mechanics (Rev. Ed.)”***
- M. A. Nielsen & I. L. Chuang (2000)
 - **“Quantum Computation and Quantum Information”**
- **量子コンピュータ授業**
 - <https://www.youtube.com/playlist?list=PLB1324F2305C028F7>
 - http://www.appi.keio.ac.jp/Ito_group/abe/

講義内容

- スピンと量子ビット
- 量子力学の公理と量子ゲート
- 量子テレポーテーション

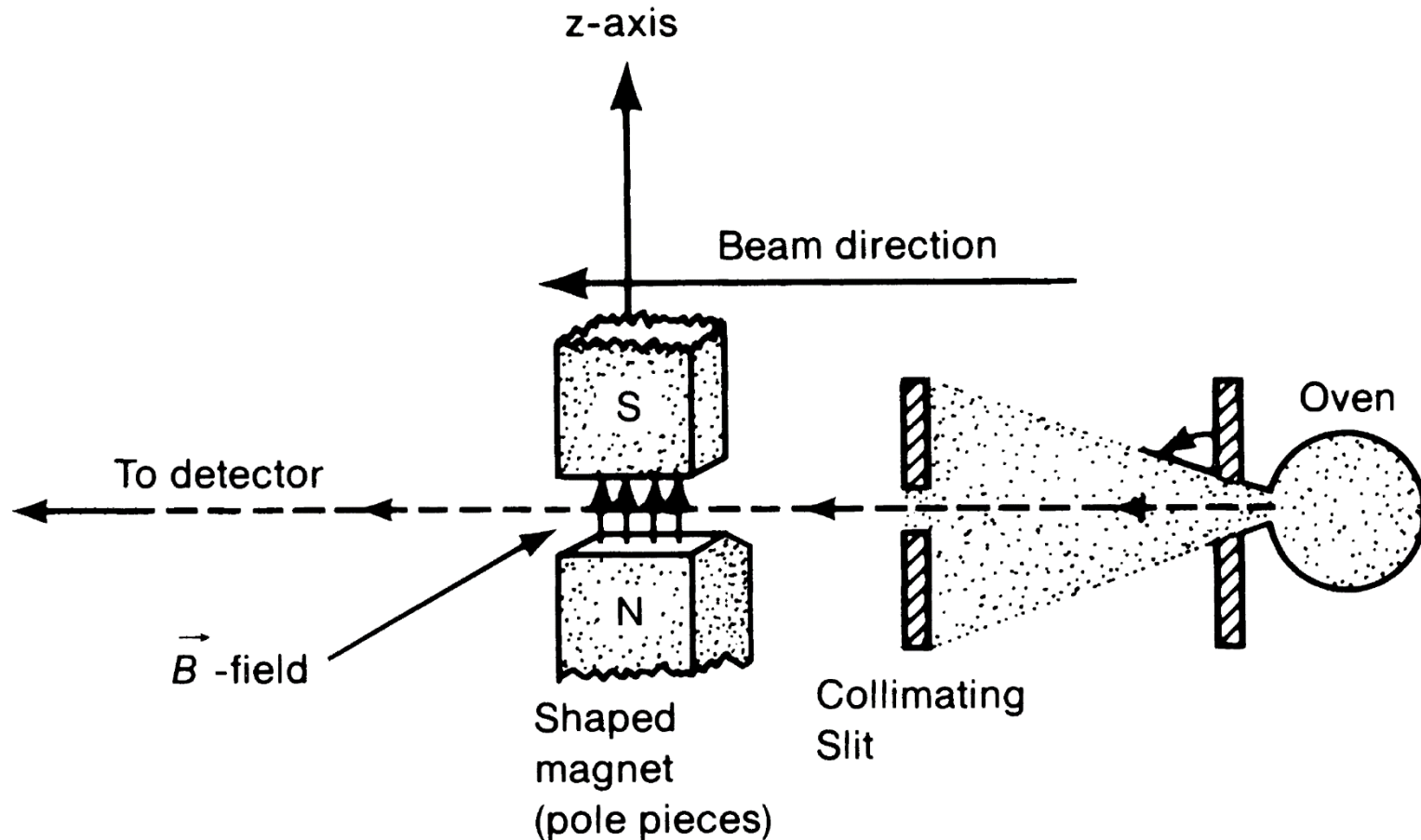
講義内容

- **スピンと量子ビット**
- 量子力学の公理と量子ゲート
- 量子テレポーテーション

シュテルン-ゲルラッハの実験

実質的にスピンを観測した実験(1922)

→ スピンの提案は1925年(G. E. Uhlenbeck & S. Goudsmit)

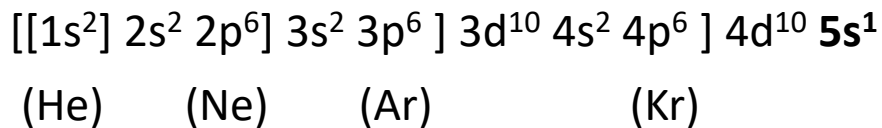
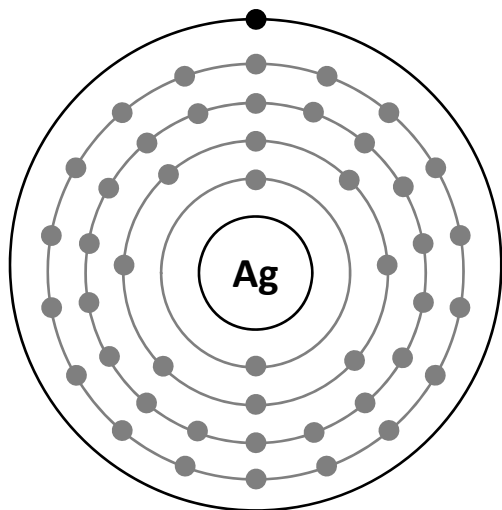


シュテルン-ゲルラッハの実験

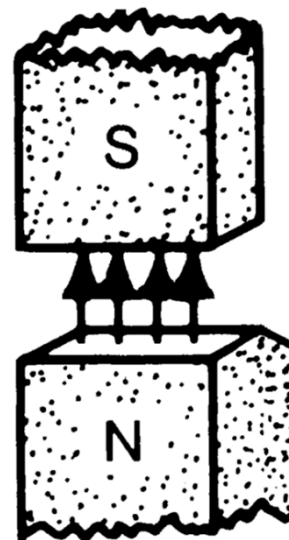
銀原子(元素番号47)

磁場勾配

磁気モーメント $\mu \propto S$



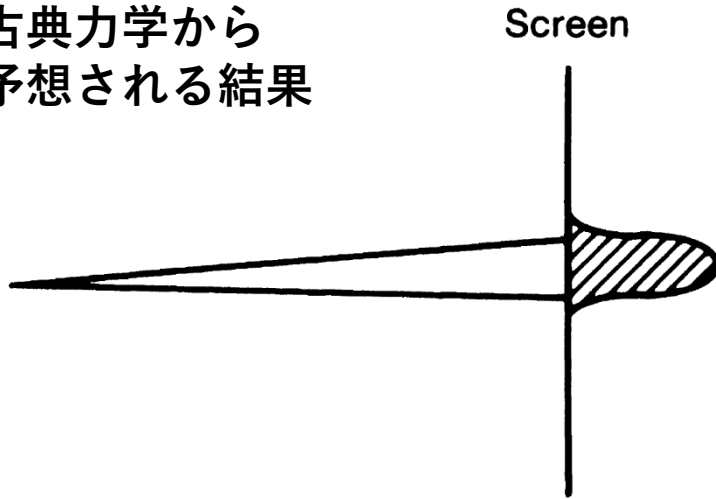
$$F_z = -\frac{\partial}{\partial z} (-\mu \cdot B) \approx \mu_z \frac{\partial B_z}{\partial z}$$



$$F = -\nabla U$$

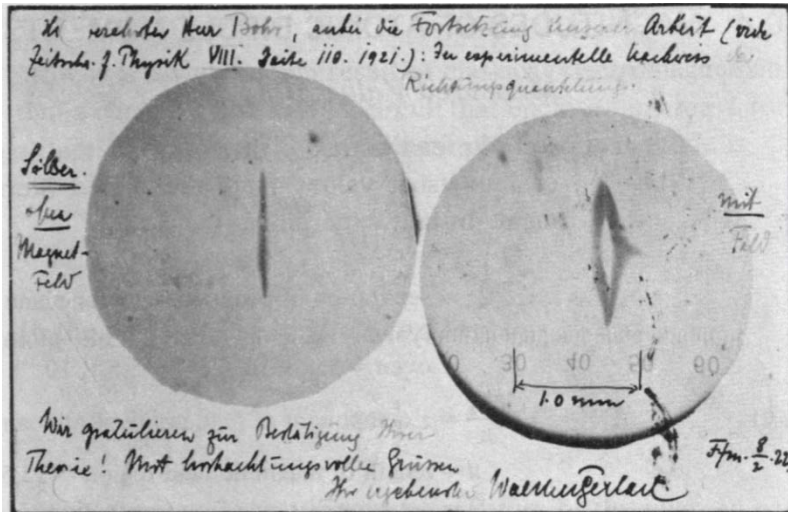
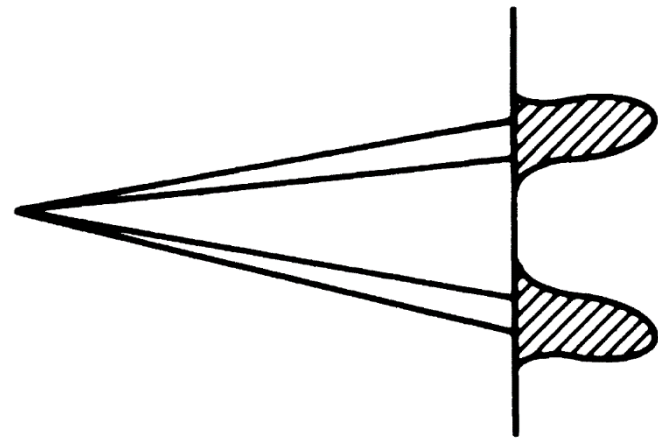
シュテルン-ゲルラッハの実験

古典力学から
予想される結果



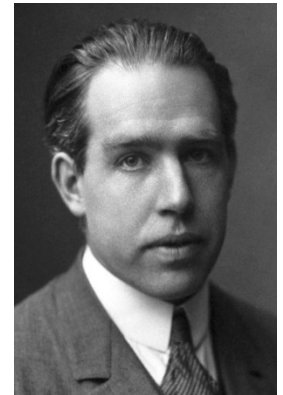
実験結果

Screen



GerlachからBohrへのハガキ
1922年2月8日付

“Attached [is] the experimental proof of
directional quantization. We congratulate
[you] on the confirmation of your theory”



Niels Bohr
(1885–1962)

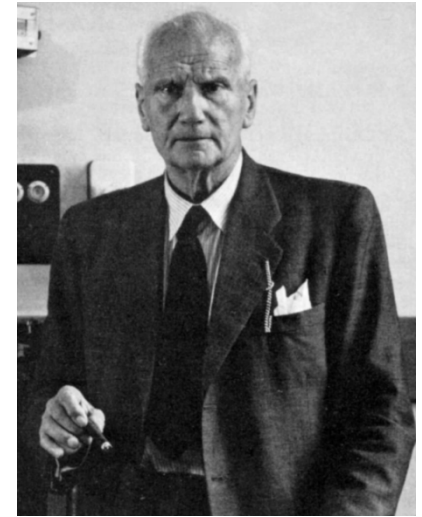
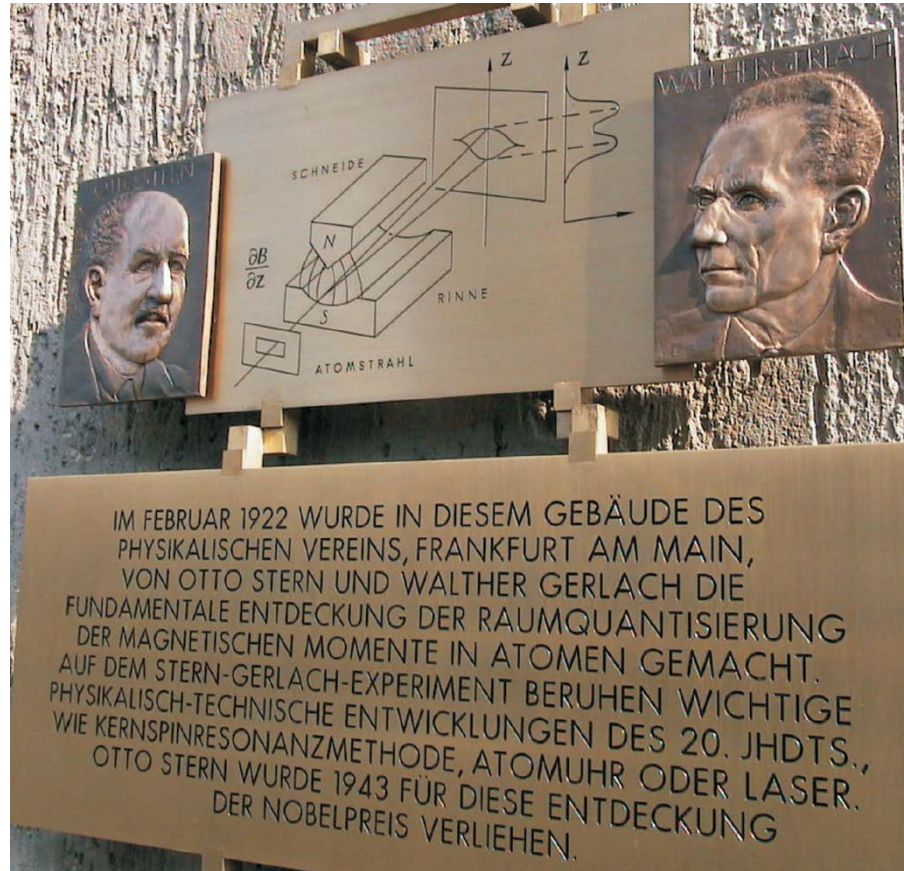
©Nobel Foundation

シュテルン-ゲルラッハの実験



Otto Stern
(1888–1969)

©P. Toschek



Walther Gerlach
(1889–1979)

©W. Schütz

@Frankfurt institute

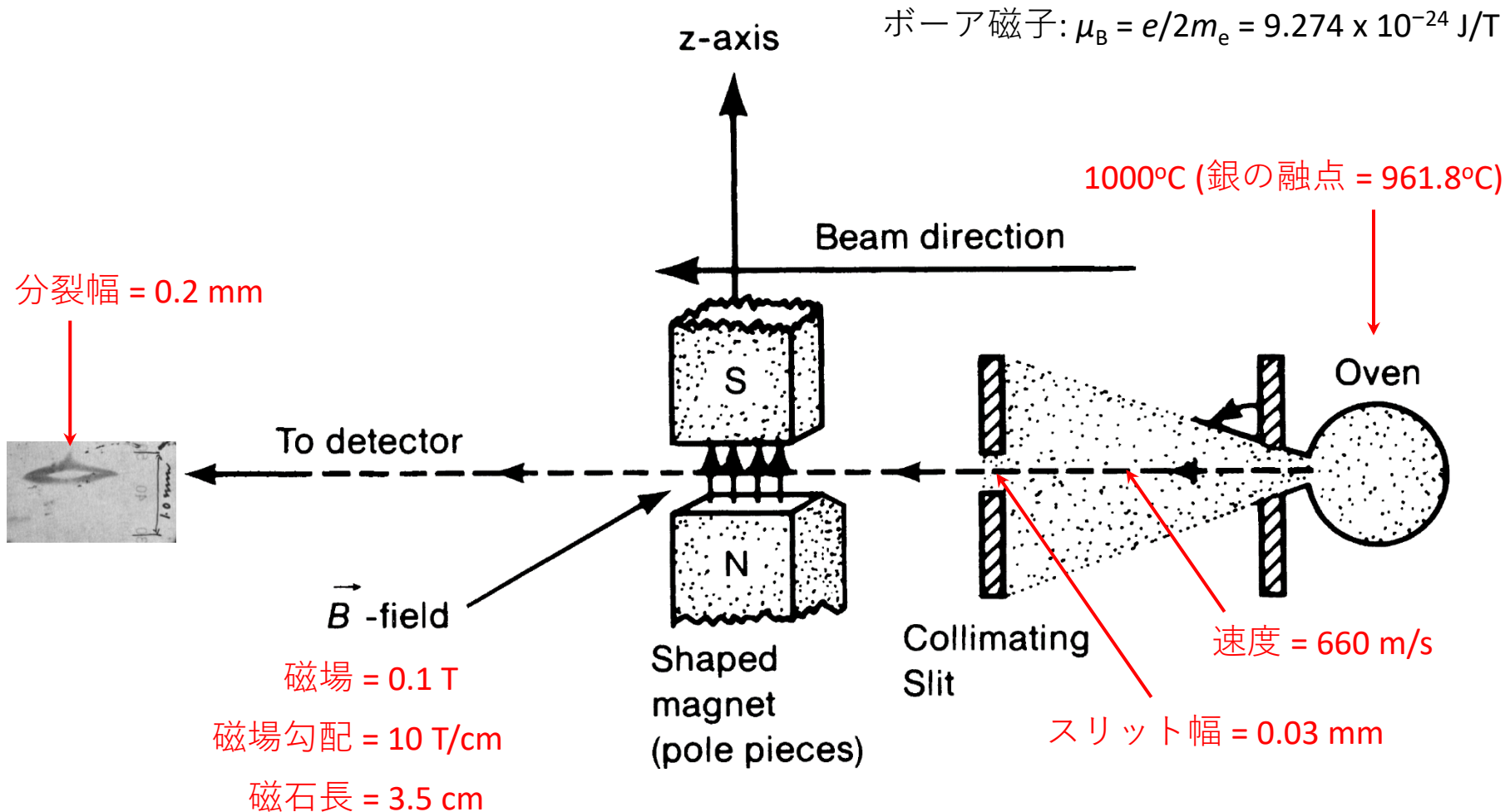
©H. Schmidt-Boöcking

“In February 1922... was made **the fundamental discovery of space quantization of the magnetic moments of atoms**. The Stern–Gerlach experiment is the basis of important scientific and technological developments in the 20th century, such as nuclear magnetic resonance, atomic clocks, or lasers...”

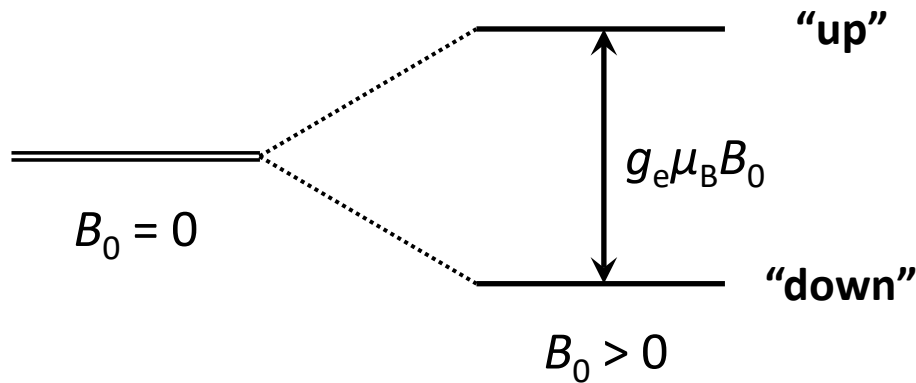
シュテルン-ゲルラッハの実験

そもそもの目的は、ボーアモデルにおける“軌道角運動量の量子化”の検証 (分裂幅 $\propto 2\mu_B$)

ボーア磁子: $\mu_B = e\hbar/2m_e = 9.274 \times 10^{-24} \text{ J/T}$

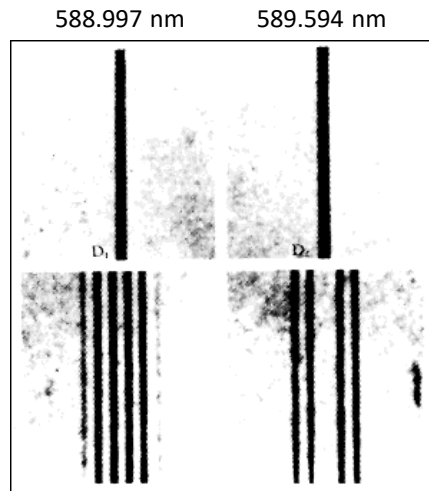


ゼーマン効果

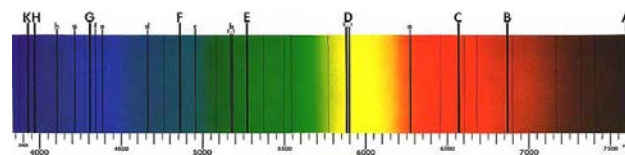


g -因子: $g_e = 2.002319$

ボーア磁子: $\mu_B = 9.274 \times 10^{-24} \text{ J/T}$



(from Wikipedia)



(from Wikipedia)

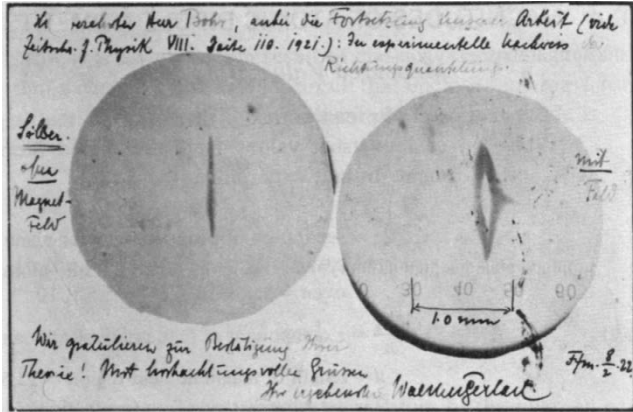
ナトリウム D_1 , D_2 線のゼーマン分裂 (1896)
→ 電子の発見 (1897, J. J. Thomson) よりも前



Pieter Zeeman
(1865–1943)

©Nobel Foundation

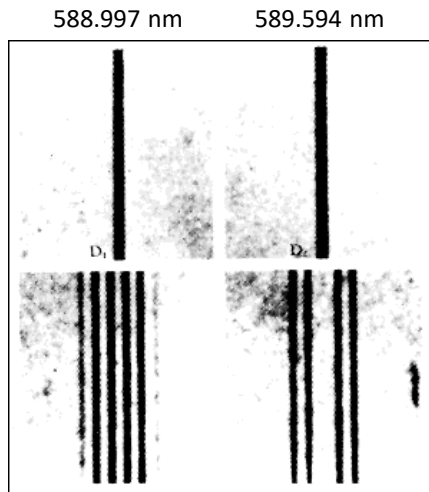
ゼーマン効果



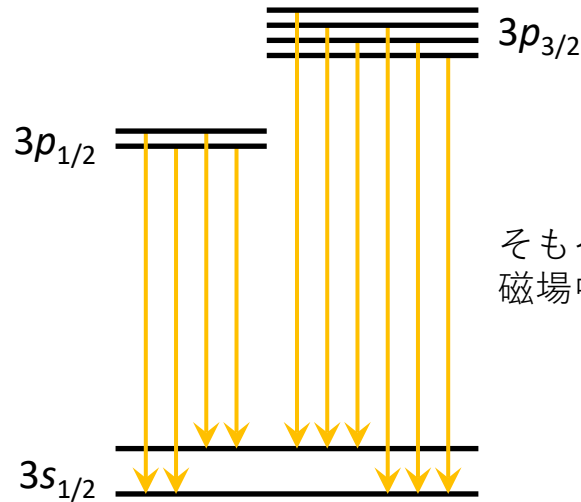
©AIP Emilio Segrè Visual Archives

スピンを考えると、“**スペクトルの分裂**”と“**空間的な分離**”を関連づけることができる...

が、スピンの性質はそれだけではない

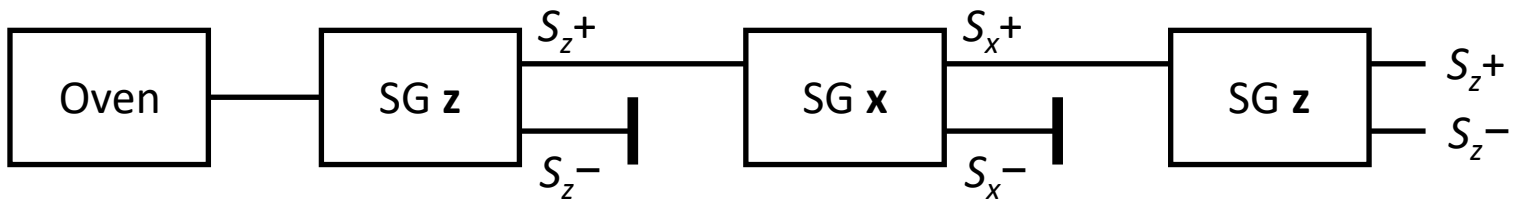
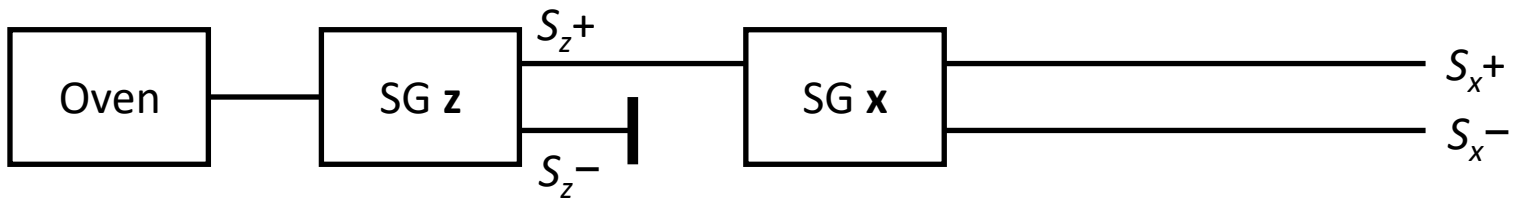
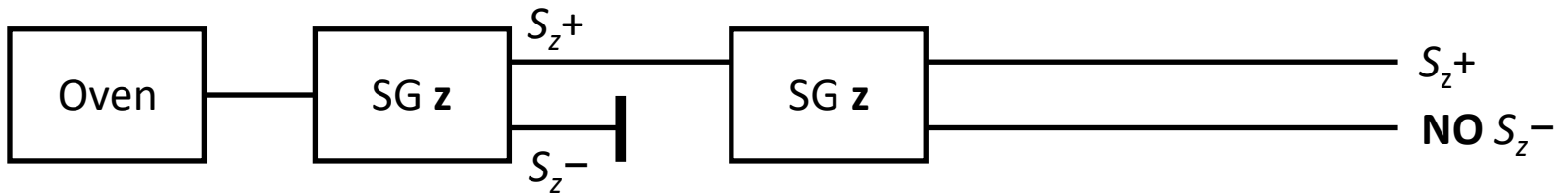


(from Wikipedia)



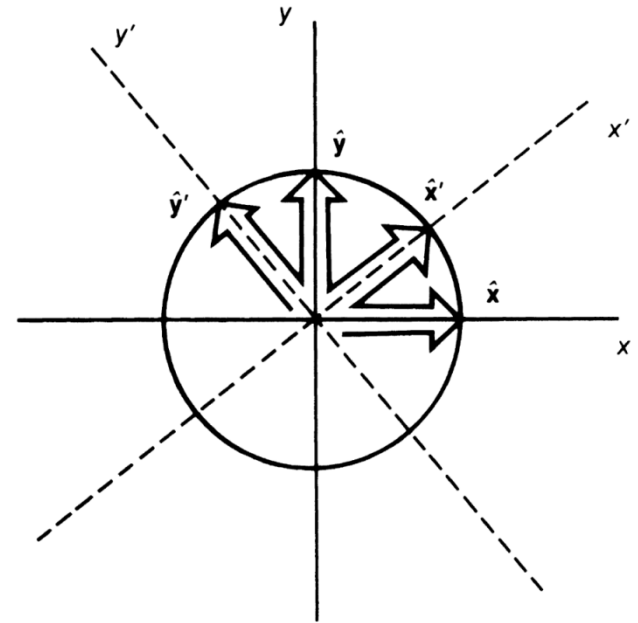
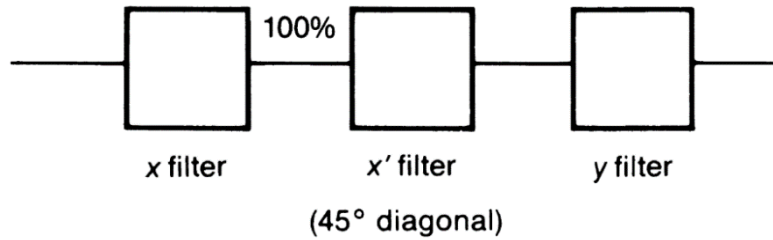
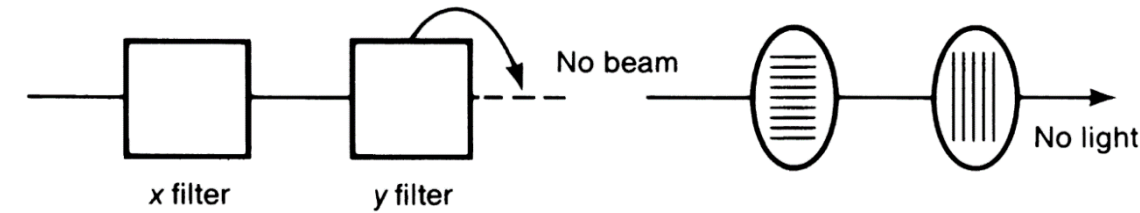
そもそも D_1 , D_2 線はスピン軌道相互作用
磁場中では計10本に分裂

連続SG実験



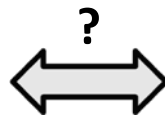
“消した”はずの s_z^- 成分が復活

光の偏光との類似性



$S_z \pm$ スピン

$S_x \pm$ スピン



x, y (水平 = H, 垂直 = V)偏光

x', y' (対角 D^+, D^-)偏光

電子スピン

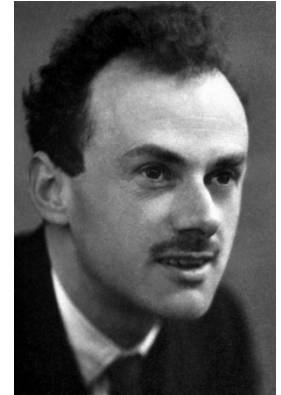
電子の持つ量子力学的角運動量(内部自由度) $S = 1/2$ ($m_s = \pm 1/2$)

角運動量としての基本的性質

$$|S_z; +\rangle = |\uparrow\rangle \quad |S_z; -\rangle = |\downarrow\rangle$$

$$S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \quad S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$$

← ケット・ベクトル
(ディラック記法)

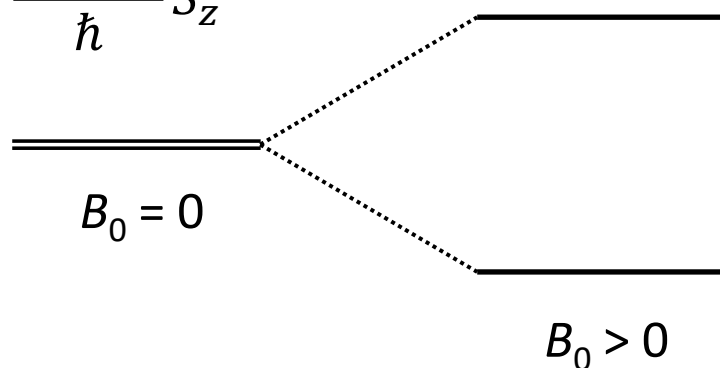


Paul Dirac
(1902–1984)

©Nobel Foundation

ハミルトニアン

$$H_Z = \frac{g\mu_B B_0}{\hbar} S_z$$



$$H_Z|\uparrow\rangle = \frac{g\mu_B B_0}{2}|\uparrow\rangle$$

$$H_Z|\downarrow\rangle = -\frac{g\mu_B B_0}{2}|\downarrow\rangle$$

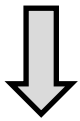
電子スピン

行列表示

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$$

$$S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$$

s_x 成分をどう表すか...?

$$S_x|S_x; +\rangle = \frac{\hbar}{2}|S_x; +\rangle$$

$$S_x|S_x; -\rangle = -\frac{\hbar}{2}|S_x; -\rangle$$

となつて欲しい

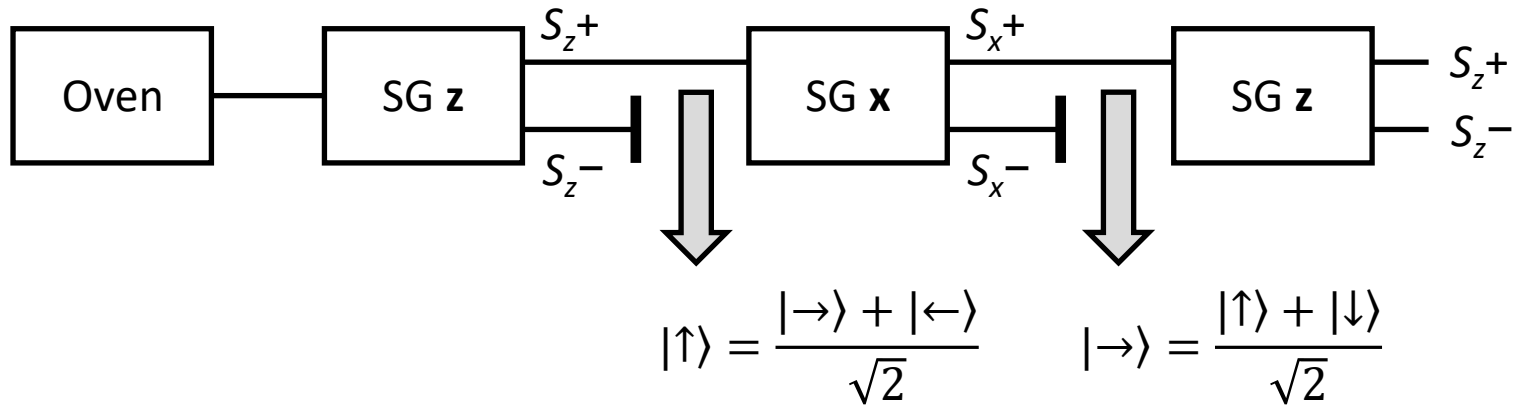


$$|S_x; +\rangle = |\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|S_x; -\rangle = |\leftarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

連続SG実験



規格化条件・直交性

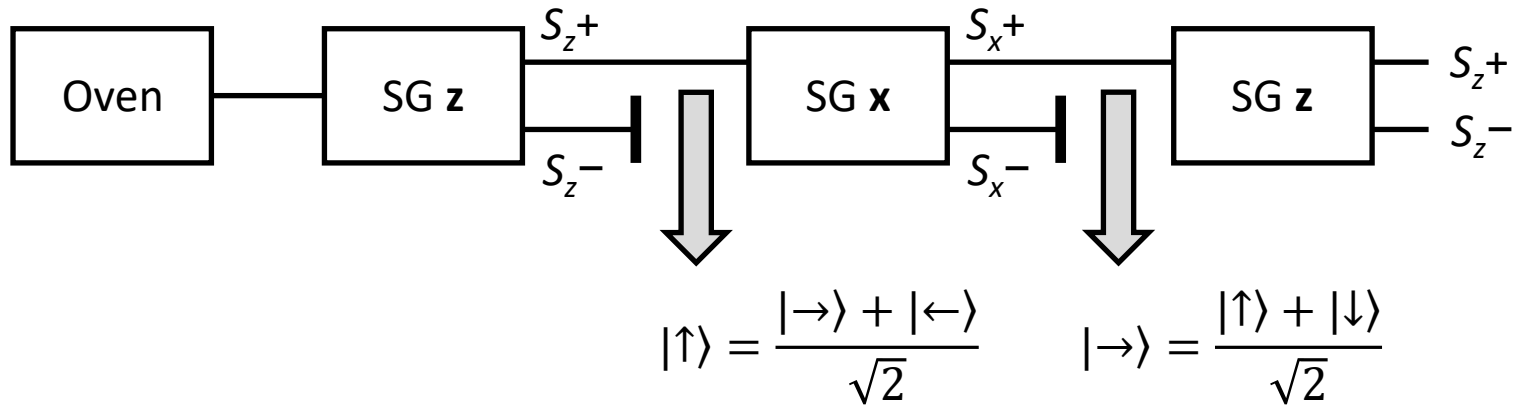
$$\begin{aligned} \langle \uparrow | &= (1 \quad 0) \\ \langle \downarrow | &= (0 \quad 1) \end{aligned} \quad \leftarrow \begin{array}{l} \text{ブラ・ベクトル} \\ \text{(ケットの転置)} \end{array}$$

$$\begin{aligned} \langle \uparrow | \uparrow \rangle &= (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \\ \langle \downarrow | \uparrow \rangle &= (0 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \end{aligned} \quad \leftarrow \begin{array}{l} \text{ブラケット} \\ \text{(括弧、内積)} \end{array}$$

注: $\frac{|\rightarrow\rangle}{\sqrt{2}}$ とはならない

$$\begin{aligned} \langle \rightarrow | \rightarrow \rangle &= \frac{1}{\sqrt{2}} (1 \quad 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \\ \langle \leftarrow | \rightarrow \rangle &= \frac{1}{\sqrt{2}} (1 \quad -1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \end{aligned}$$

連続SG実験



“スピンのz成分とx成分は独立の物理量ではない”

“スピンのz成分とx成分を同時に決定することはできない”

“演算子が交換しない”

$$S_z S_x - S_x S_z = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

パウリ行列

固有ベクトル

(固有値 = 1)

(固有値 = -1)

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\leftarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|\rightarrow_y\rangle = \frac{|\uparrow\rangle + i|\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\leftarrow_y\rangle = \frac{|\uparrow\rangle - i|\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

交換関係等

$$[\sigma_i, \sigma_{i+1}] = \sigma_i \sigma_{i+1} - \sigma_{i+1} \sigma_i = 2i \sigma_{i+2}$$

$$\{\sigma_i, \sigma_{i+1}\} = \sigma_i \sigma_{i+1} + \sigma_{i+1} \sigma_i = 0$$

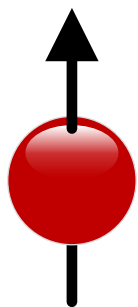
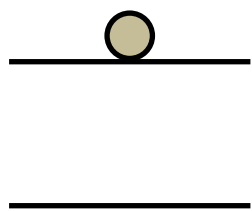
$$\sigma_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I$$



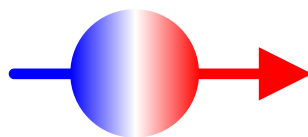
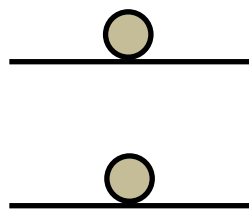
Wolfgang Pauli
(1900–1958)

©Nobel Foundation

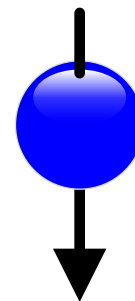
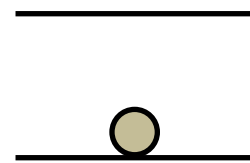
量子ビット = スピン1/2



$$|0\rangle \equiv |\uparrow\rangle$$



$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



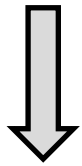
$$|1\rangle \equiv |\downarrow\rangle$$

重ね合わせ

量子ビット

定義: 計算基底のベクトル表示

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \langle 0|0\rangle = (1 \quad 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$



$$\langle 1|0\rangle = (0 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

公理(Postulate): 許される状態はヒルベルト空間内

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

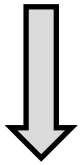
$$|\alpha|^2 + |\beta|^2 = 1 \quad \alpha, \beta \in \mathbf{C}$$

$$\langle \psi|\psi\rangle = (\alpha^* \quad \beta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1$$

量子ビット

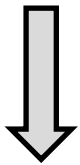
重ね合わせ状態

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$



$$|\psi\rangle = \underline{e^{i\gamma}} \left(\cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \right) \quad \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \gamma, \phi < 2\pi \end{array}$$

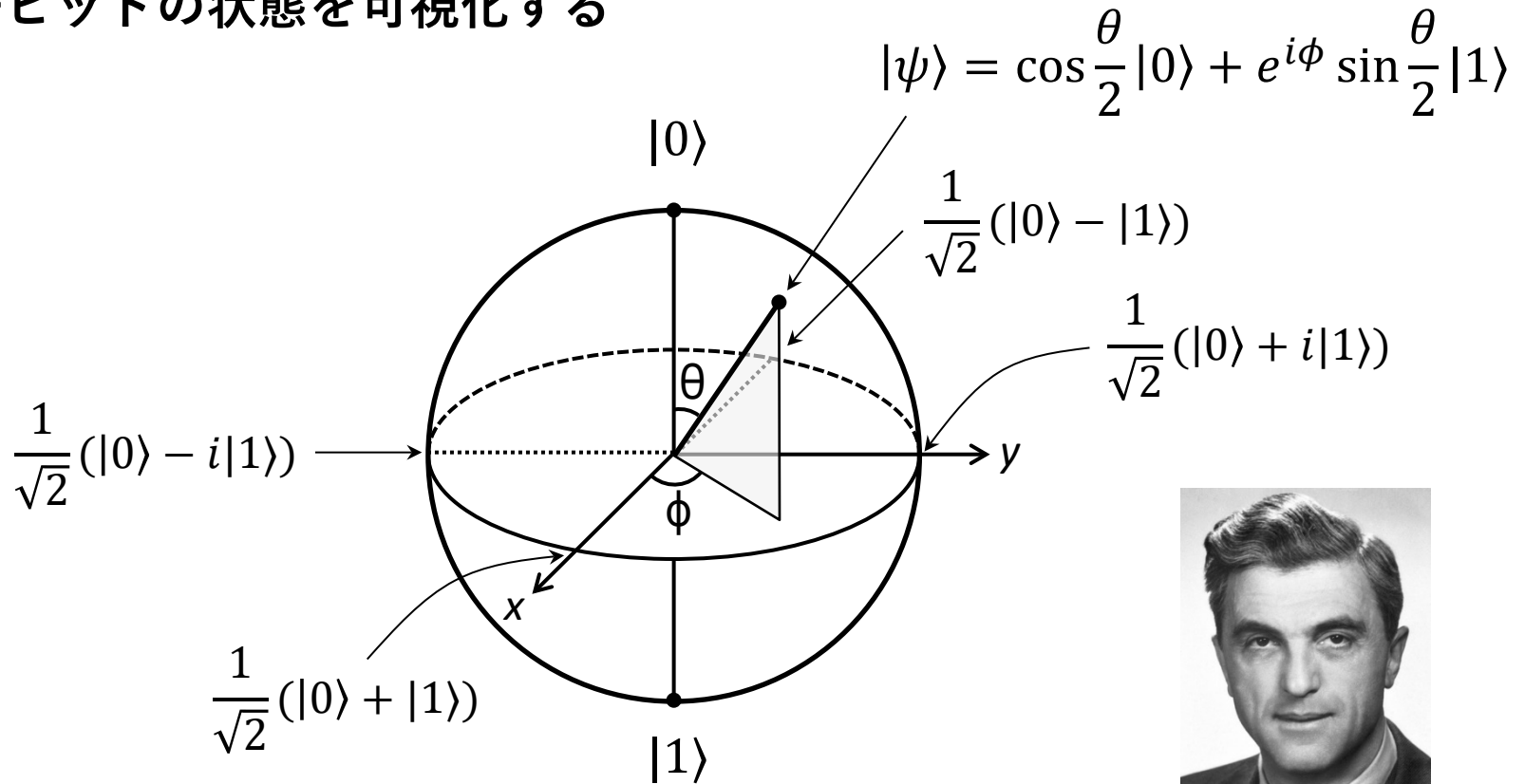
測定に影響しない



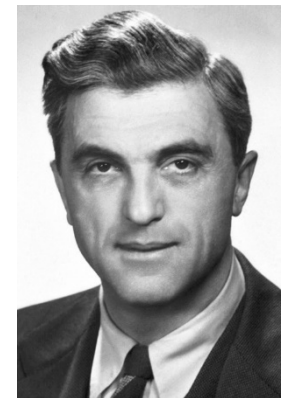
$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \quad \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi \end{array}$$

ブロッホ球

量子ビットの状態を可視化する



x, y, z 軸上の点がそれぞれ $\sigma_{x,y,z}$ の固有状態に対応

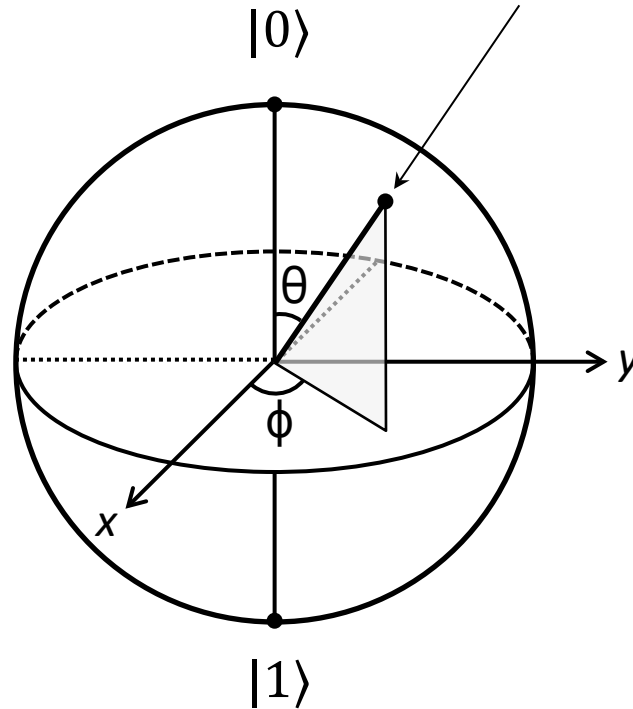


Felix Bloch
(1905–1983)

ブロッホ球

$|\psi\rangle$ を固有状態とする演算子は?

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



予想

$$\boldsymbol{\psi} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} = (\sin\theta \cos\phi)\sigma_x + (\sin\theta \sin\phi)\sigma_y + \cos\theta \sigma_z$$

ブロッホ球

$|\psi\rangle$ を固有状態とする演算子は? (予想)

$$\boldsymbol{\psi} \cdot \boldsymbol{\sigma} = (\sin \theta \cos \phi) \sigma_x + (\sin \theta \sin \phi) \sigma_y + \cos \theta \sigma_z$$

$$= \sin \theta \cos \phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin \theta \sin \phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}$$

レポート課題・第1問(10点)

$|\psi\rangle$ が演算子 $\boldsymbol{\psi} \cdot \boldsymbol{\sigma}$ の固有状態で、固有値 = 1であることを確認せよ。

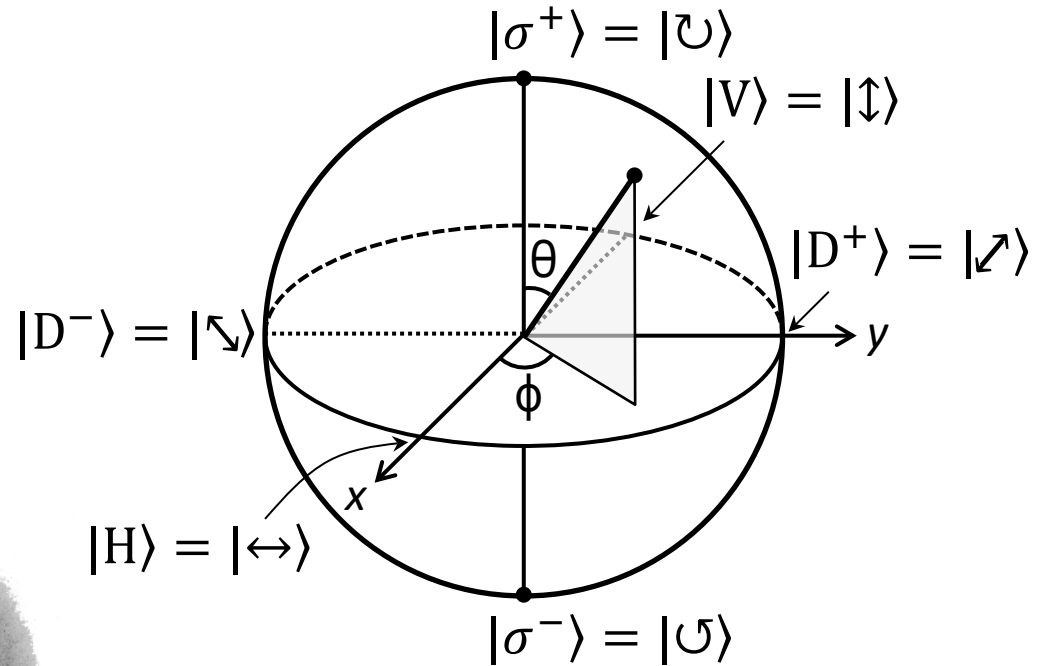
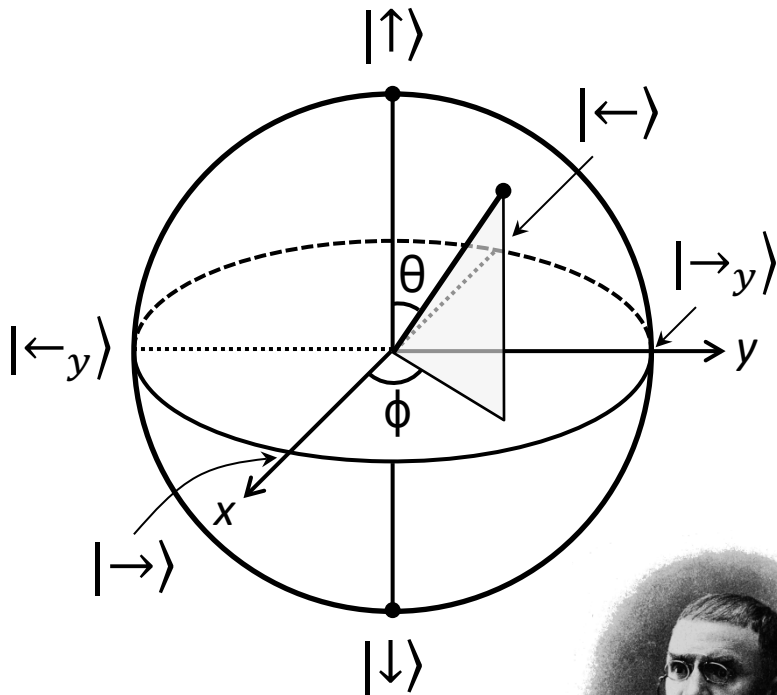
さらに、固有値 = -1となる状態 $|\psi'\rangle$ を見つけそのことを確認せよ。

ブロッホ球とポアンカレ球

ブロッホ球
(スピン)



ポアンカレ球
(光の偏光)



Henri Poincaré
(1854–1912)
(from Wikipedia)

講義内容

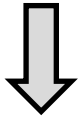
- スピンと量子ビット
- **量子力学の公理と量子ゲート**
- 量子テレポーテーション

ユニタリ発展

定義: エルミート共役 $A^\dagger = (A^T)^*$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

定義: 自己共役 $A = A^\dagger$

定義: ユニタリ $UU^\dagger = I$

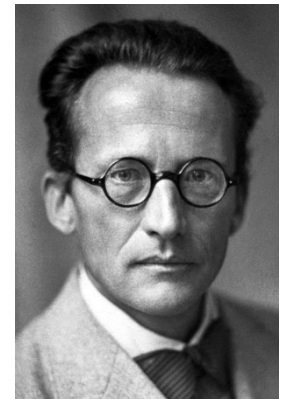


公理: 量子状態の時間発展はユニタリ

シュレディンガー方程式の解

$$|\Psi(t + \Delta t)\rangle = \exp\left(-\frac{iH\Delta t}{\hbar}\right) |\Psi(t)\rangle$$

ハミルトニアン H を指数演算子化した e^{-iH} はユニタリ



Erwin Schrödinger
(1887–1961)

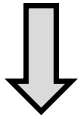
©Nobel Foundation

ユニタリ発展

定義: エルミート共役 $A^\dagger = (A^T)^*$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

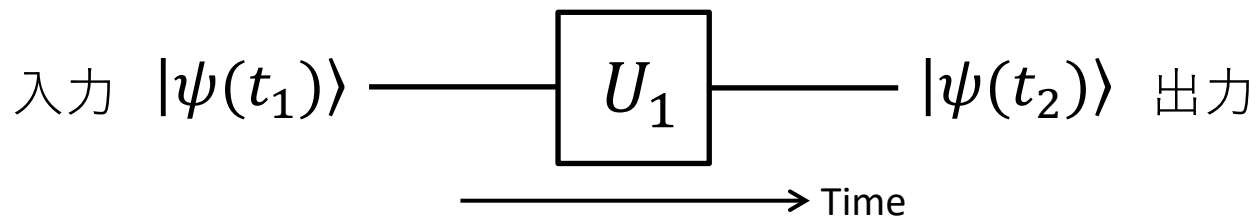
定義: 自己共役 $A = A^\dagger$

定義: ユニタリ $UU^\dagger = I$



公理: 量子状態の時間発展はユニタリ

$$|\psi(t_2)\rangle = U_1 |\psi(t_1)\rangle$$

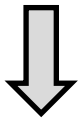


ユニタリ発展

定義: エルミート共役 $A^\dagger = (A^T)^*$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

定義: 自己共役 $A = A^\dagger$

定義: ユニタリ $UU^\dagger = I$

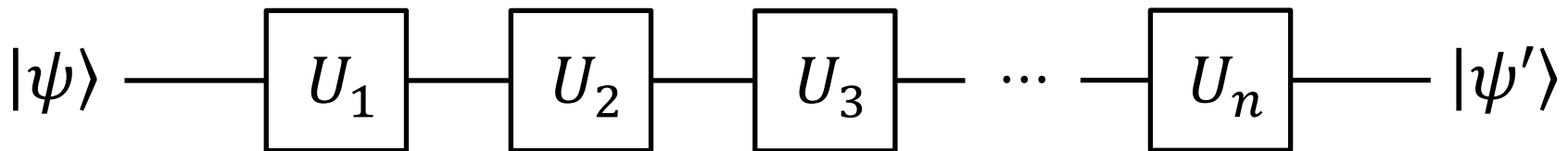


公理: 量子状態の時間発展はユニタリ

$$|\psi(t_2)\rangle = U_1 |\psi(t_1)\rangle$$

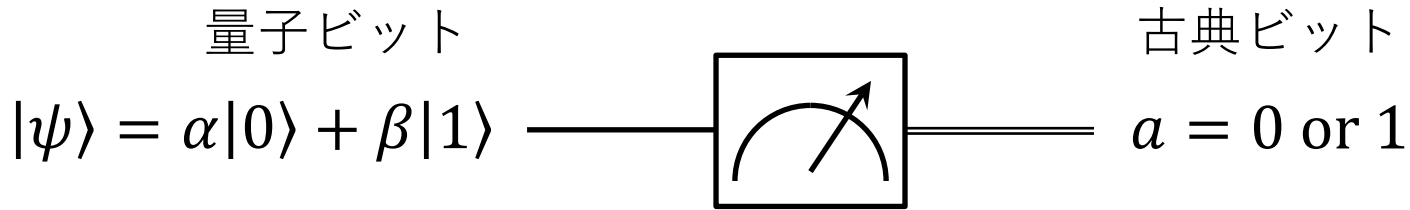
$$U_n \cdots U_2 U_1 |\psi\rangle$$

||



1量子ビットゲート列(量子回路)

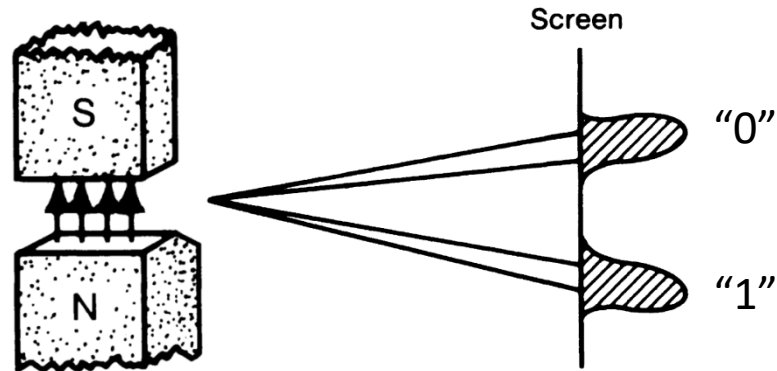
測定



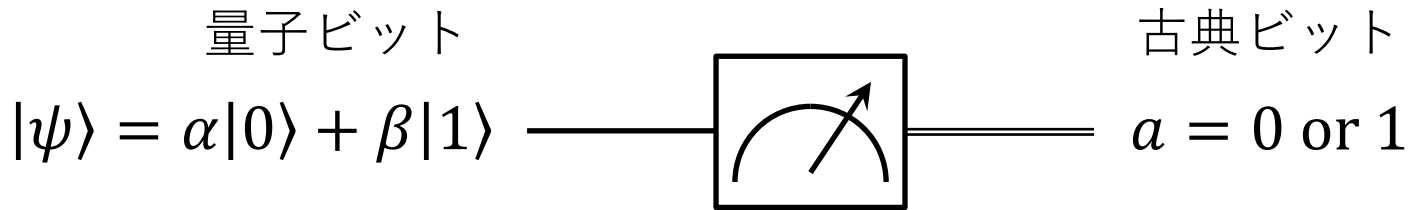
計算基底(z基底)による測定を行う測定器

→ 確率 $|\alpha|^2$ で“0”、確率 $|\beta|^2$ で“1”を出力

固有値との対応: $(-1)^a$



測定



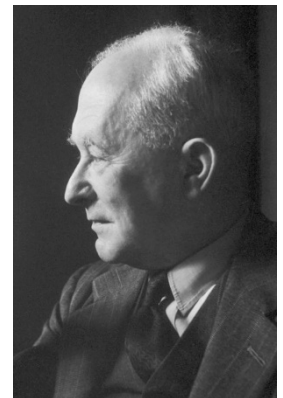
計算基底(z基底)による測定を行う測定器

→ 確率 $|\alpha|^2$ で“0”、確率 $|\beta|^2$ で“1”を出力

公理: 射影測定と確率解釈

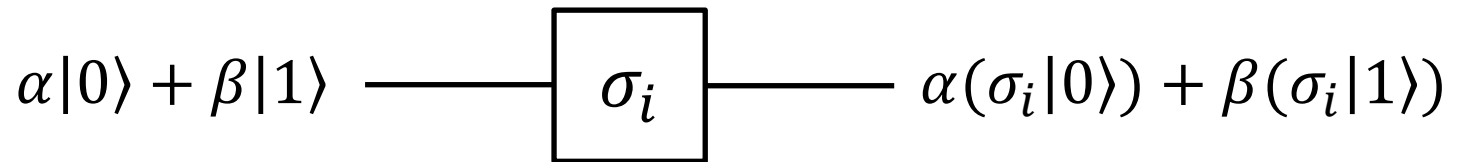
$$\left\{ \begin{array}{l} p_0 = \langle \psi | 0 \rangle \langle 0 | \psi \rangle \\ \quad = (\alpha^* \langle 0 | 0 \rangle + \beta^* \langle 1 | 0 \rangle) (\alpha \langle 0 | 0 \rangle + \beta \langle 0 | 1 \rangle) = |\alpha|^2 \\ p_1 = \langle \psi | 1 \rangle \langle 1 | \psi \rangle = |\langle 1 | \psi \rangle|^2 = |\beta|^2 \end{array} \right.$$

本講義では測定理論の詳細には立ち入りません
とりあえず、係数の2乗で確率が出るとっておけばOK



Max Born
(1882–1970)

パウリゲート



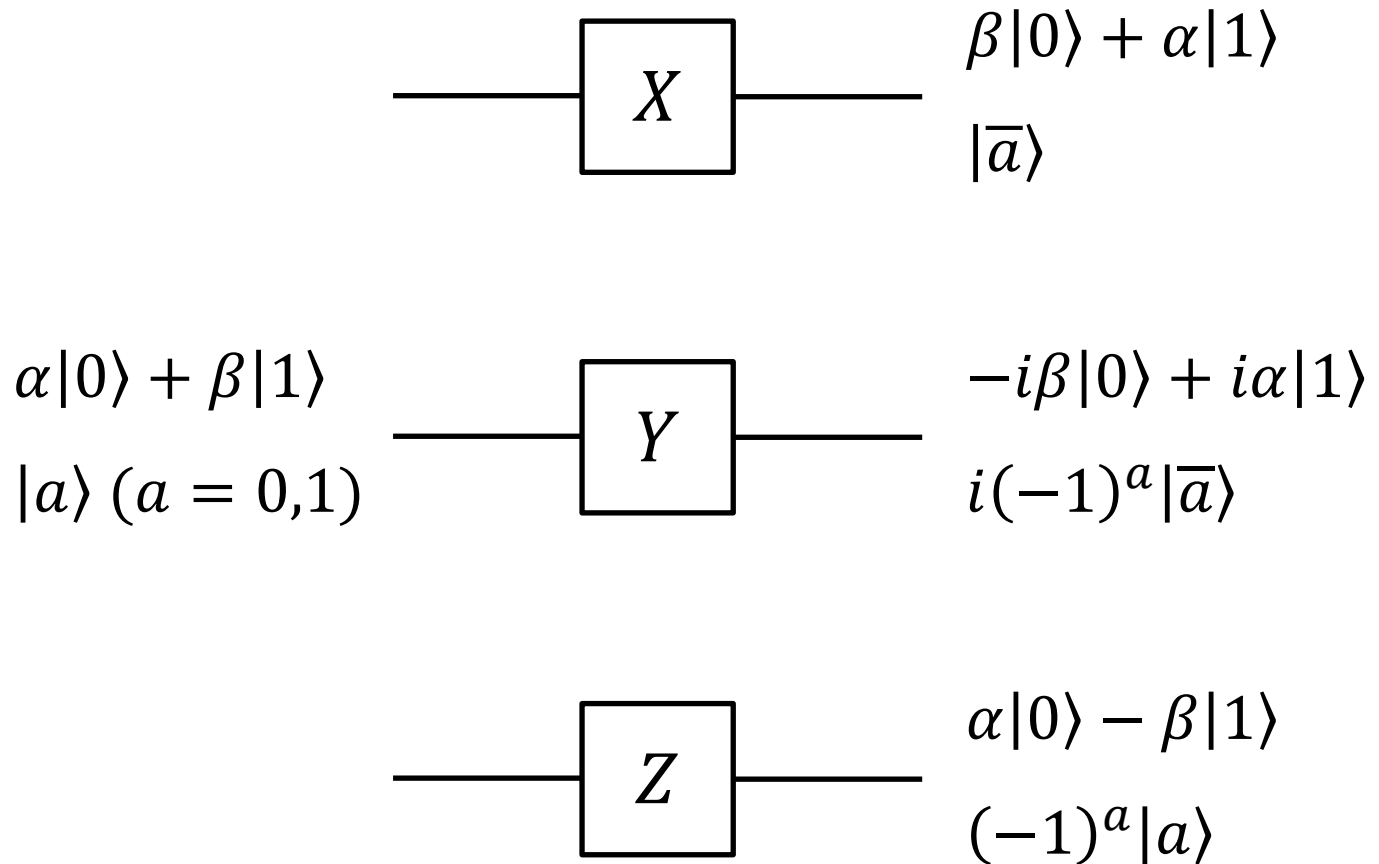
パウリ行列

$$\sigma_1 = \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$\sigma_2 = \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = i \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$$

$$\sigma_3 = \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

パウリゲート

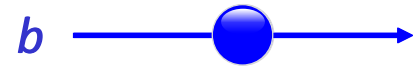


経路量子ビット

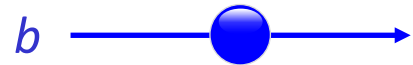
単一光子のいる経路に量子情報をエンコードする



$$|1\rangle_a |0\rangle_b \equiv |0\rangle_L$$



$$|0\rangle_a |1\rangle_b \equiv |1\rangle_L$$



$$\alpha|0\rangle_L + \beta|1\rangle_L$$

マッハーツェンダー干渉計

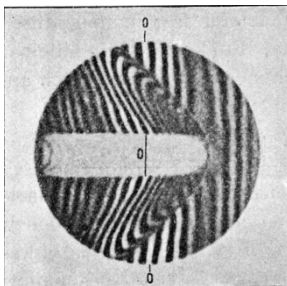
Ludwig Mach
(1868–1951)

父子

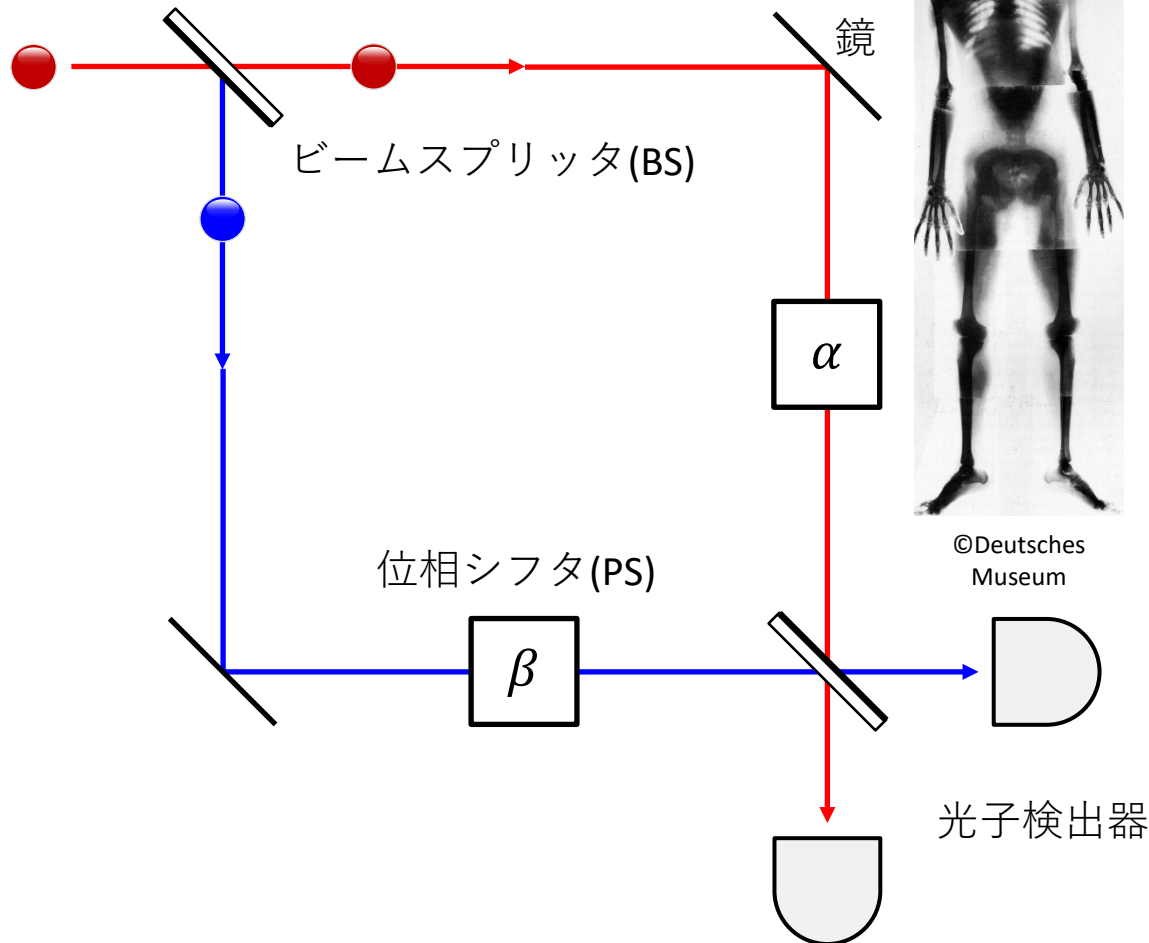


Ernst Mach
(1838–1916)

(from Wikipedia)

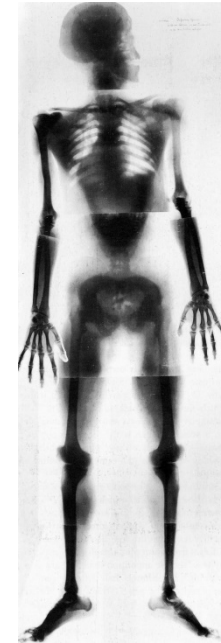


(from Stud. Hist. Philos. Sci. A **44**, 1 (2013) Hoffmann)



Ludwig Zehnder
(1854–1949)

子弟



©Deutsches
Museum

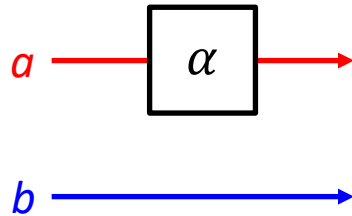


Wilhelm Röntgen
(1845–1923)

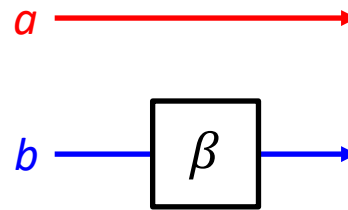
©Nobel Foundation

量子ゲートとしての光学素子

位相シフタ(行路差)

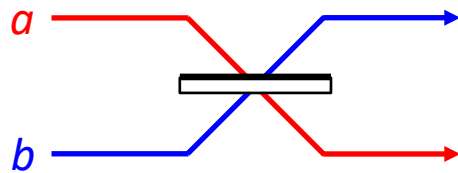


$$P_a = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$



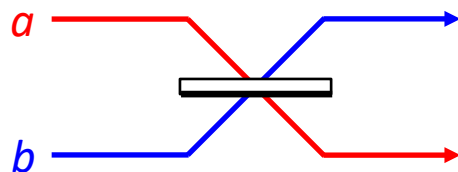
$$P_b = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}$$

ビームスプリッタ



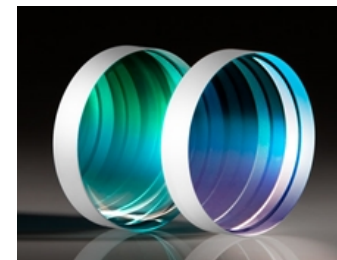
$$B = \begin{pmatrix} \sqrt{T} & \sqrt{R} \\ -\sqrt{R} & \sqrt{T} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{aligned} T &= \frac{1}{2} \text{ (透過率)} \\ R &= \frac{1}{2} \text{ (反射率)} \\ T + R &= 1 \end{aligned}$$



$$B^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$n_{\text{air}} < n_{\text{coat}} < n_{\text{glass}}$$



マッハツェンダー干渉計

時間発展

$$|0\rangle_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow B|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow P_b P_a B|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\alpha} \\ -e^{i\beta} \end{pmatrix}$$

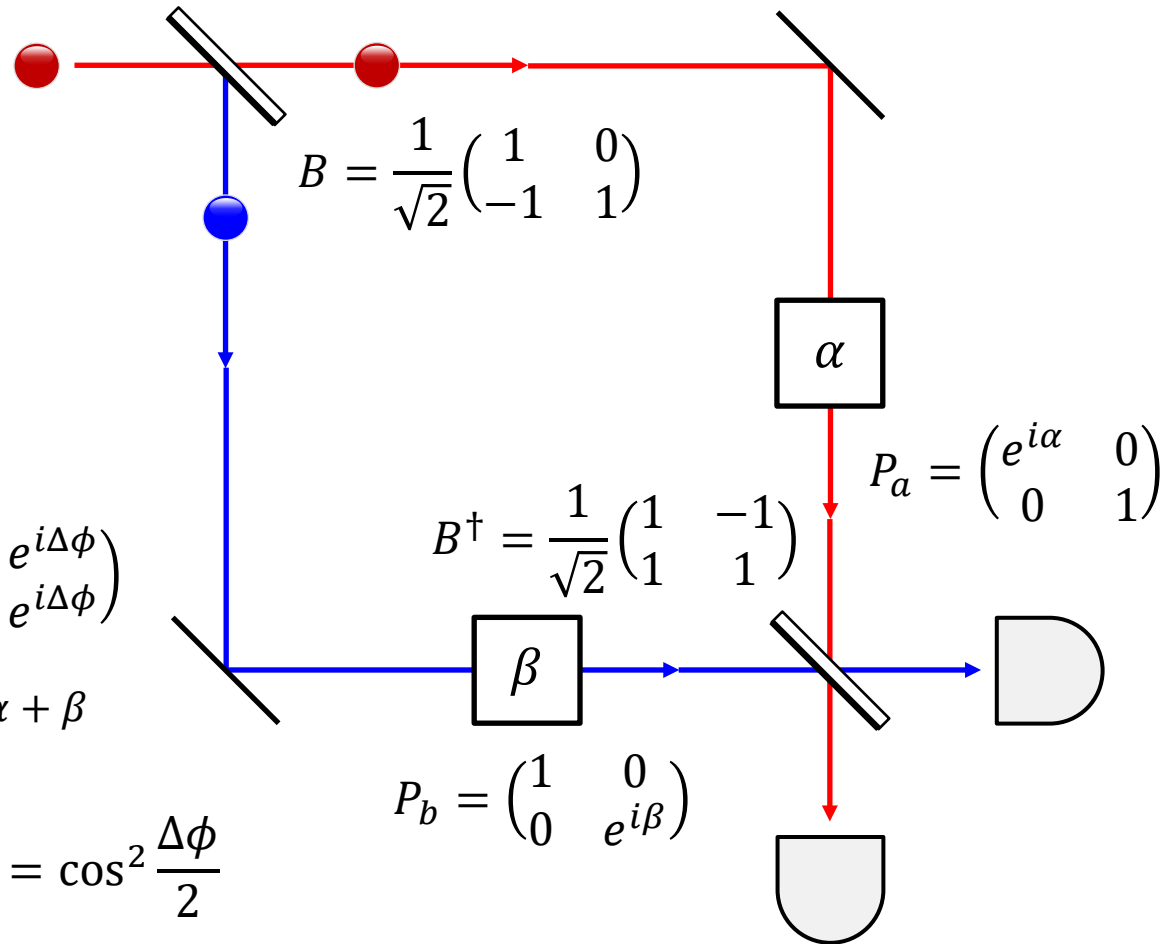
$$\Rightarrow B^\dagger P_b P_a B|0\rangle = \frac{e^{i\alpha}}{2} \begin{pmatrix} 1 + e^{i\Delta\phi} \\ 1 - e^{i\Delta\phi} \end{pmatrix}$$

$$\Delta\phi = -\alpha + \beta$$

測定

$$D_a = |\langle 0 | B^\dagger P_b P_a B | 0 \rangle|^2 = \cos^2 \frac{\Delta\phi}{2}$$

$$D_b = |\langle 1 | B^\dagger P_b P_a B | 0 \rangle|^2 = \sin^2 \frac{\Delta\phi}{2}$$



アダマールゲート

$$|a\rangle \longrightarrow \boxed{H} \longrightarrow \frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} |b\rangle = \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$$

$$\left\{ \begin{array}{l} H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ H \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array} \right\} \longleftrightarrow H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$HH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\longleftrightarrow H^\dagger = H \quad (\text{自己共役})$$

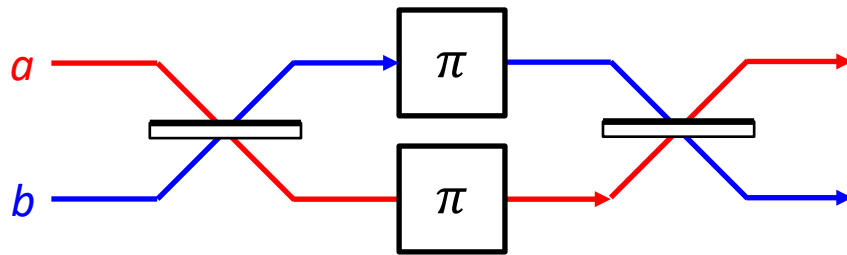
Jacques Hadamard
(1865–1965)
(from Wikipedia)



アダマールゲート

$$|a\rangle \longrightarrow \boxed{H} \longrightarrow \frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} |b\rangle = \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$$

MZ干渉計($\alpha = \beta = \pi, \Delta\phi = 0$)



$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P_a B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \approx H$$

$$B^\dagger P_b = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

アダマールゲート

$$|a\rangle \longrightarrow \boxed{H} \longrightarrow \frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} |b\rangle = \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$$

$$HH|a\rangle = H \left(\frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} |b\rangle \right)$$

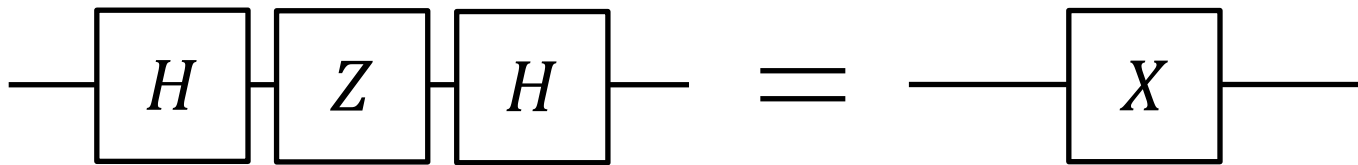
$$(a+c) \cdot b = \begin{cases} 0 & (c = a) \\ b & (c = \bar{a}) \end{cases}$$

$$= \frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} \left(\frac{1}{\sqrt{2}} \sum_{c=0,1} (-1)^{b \cdot c} |c\rangle \right) = \frac{1}{2} \sum_{b,c} (-1)^{(a+c) \cdot b} |c\rangle$$

$$= \frac{1}{2} \sum_b (|a\rangle + (-1)^b |\bar{a}\rangle) = \frac{1}{2} (|a\rangle + |\bar{a}\rangle + |a\rangle - |\bar{a}\rangle) = |a\rangle$$

干渉による強め合いと弱め合い

アダマールゲート



$$a \cdot b + b = (a + 1) \cdot b = \bar{a} \cdot b$$

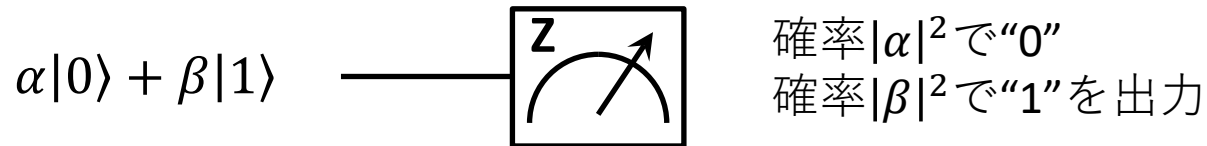
$$\begin{aligned}
 HZH|a\rangle &= HZ \left(\frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b} |b\rangle \right) = H \left(\frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{a \cdot b + b} |b\rangle \right) \\
 &= \frac{1}{\sqrt{2}} \sum_{b=0,1} (-1)^{\bar{a} \cdot b} \left(\frac{1}{\sqrt{2}} \sum_{c=0,1} (-1)^{b \cdot c} |c\rangle \right) = \frac{1}{2} \sum_{b,c} (-1)^{(\bar{a} + c) \cdot b} |c\rangle \\
 &= \frac{1}{2} \sum_b (|\bar{a}\rangle + (-1)^b |a\rangle) = \frac{1}{2} (|\bar{a}\rangle + |a\rangle + |\bar{a}\rangle - |a\rangle) = |\bar{a}\rangle
 \end{aligned}$$

干渉による強め合いと弱め合い

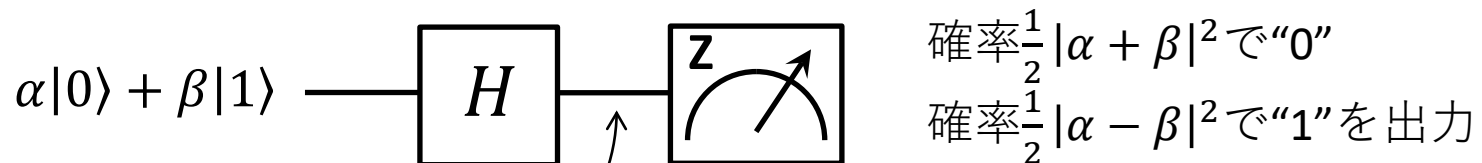
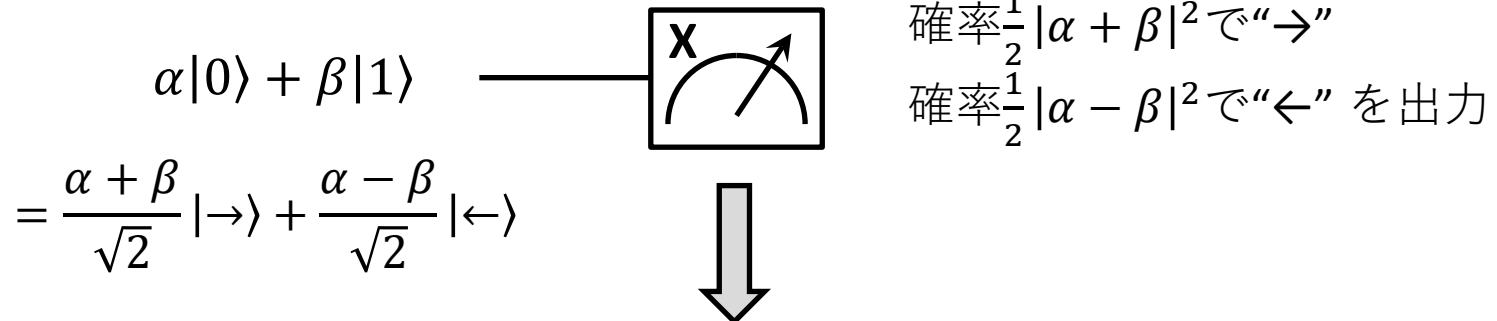
$$(\bar{a} + c) \cdot b = \begin{cases} b & (c = a) \\ 0 & (c = \bar{a}) \end{cases}$$

測定基底の変換

計算(z)基底での測定



x基底での測定



$\frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle$

“基底変換” & “計算(z)基底による測定”
で任意の基底による測定が可能

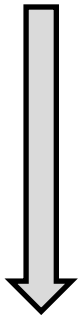
複数量子ビット

2量子ビットの状態の記述

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

公理: 複合系の状態はテンソル積で表される



(注) 2量子ビットの状態の計算基底は4つ

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

定義: テンソル積(行列表示)

$$a \otimes b = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \times \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \times \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

2量子ビット

2量子ビットの計算基底

$$|00\rangle = |0\rangle|0\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = |0\rangle|1\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

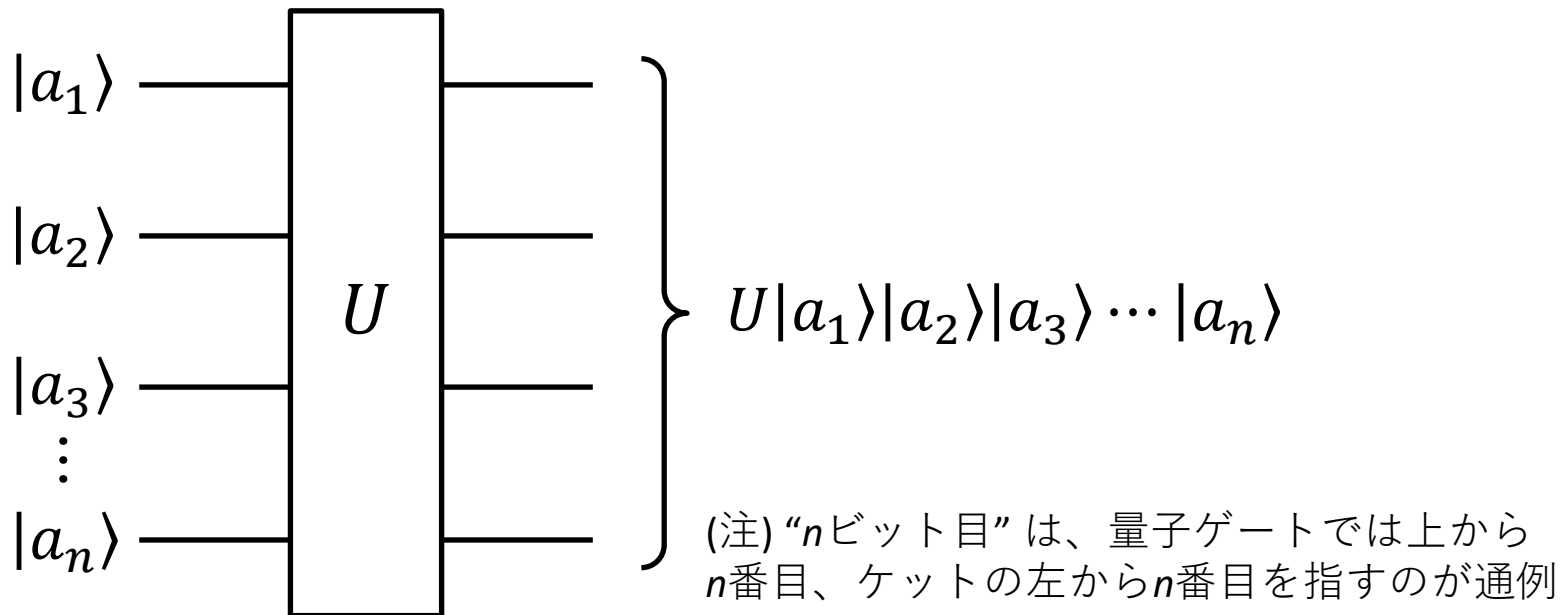
$$|10\rangle = |1\rangle|0\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle = |1\rangle|1\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2量子ビット状態

$$|\Psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{pmatrix}$$

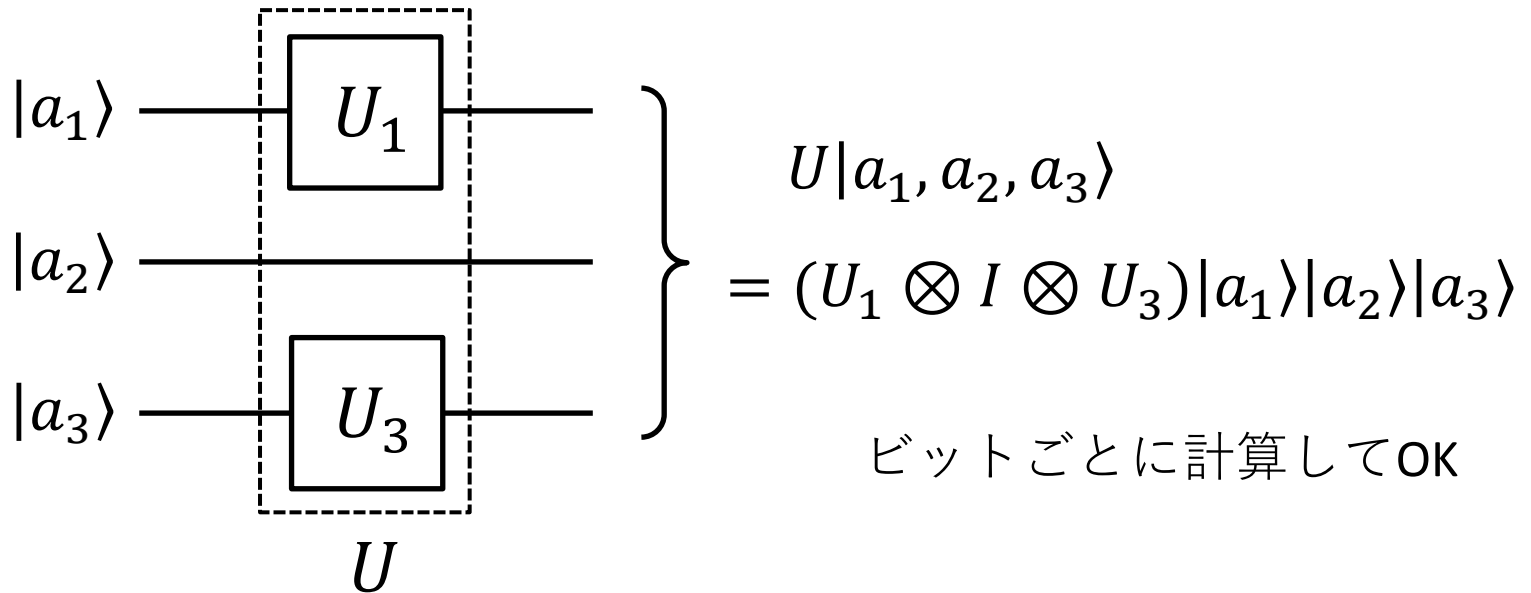
複数量子ビットゲート



$|a_1\rangle|a_2\rangle|a_3\rangle\cdots|a_n\rangle$: 2^n 次元の状態ベクトル

U : $2^n \times 2^n$ のユニタリ行列

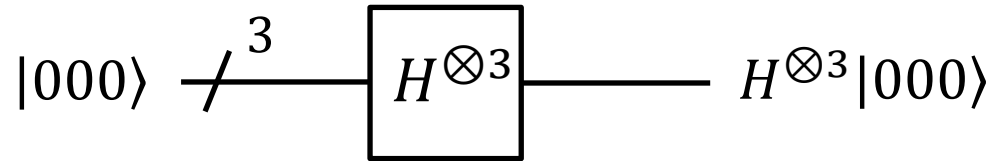
複数量子ビットゲート



定義: テンソル積(行列表示)

$$A \otimes B = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 \times B & a_3 \times B \\ a_2 \times B & a_4 \times B \end{pmatrix}$$
$$= \begin{pmatrix} a_1 b_1 & a_1 b_3 & a_3 b_1 & a_3 b_3 \\ a_1 b_2 & a_1 b_4 & a_3 b_2 & a_3 b_4 \\ a_2 b_1 & a_2 b_3 & a_4 b_1 & a_4 b_3 \\ a_2 b_2 & a_2 b_4 & a_4 b_2 & a_4 b_4 \end{pmatrix}$$

n 量子ビットの重ね合わせ

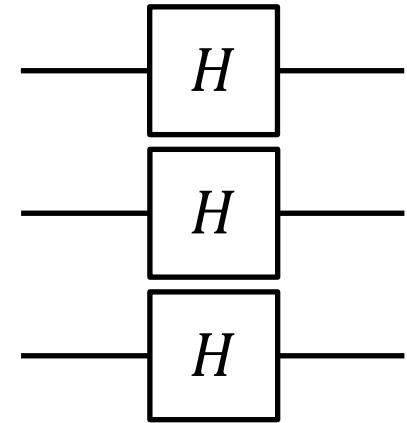


$$H^{\otimes 3}|000\rangle$$

$$= \frac{1}{\sqrt{2^3}} (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2^3}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

$$= \frac{1}{\sqrt{2^3}} \sum_{a,b,c=0,1} |abc\rangle = \frac{1}{\sqrt{2^3}} \sum_{x=0}^{2^3-1} |x\rangle$$



n 量子ビットの重ね合わせ

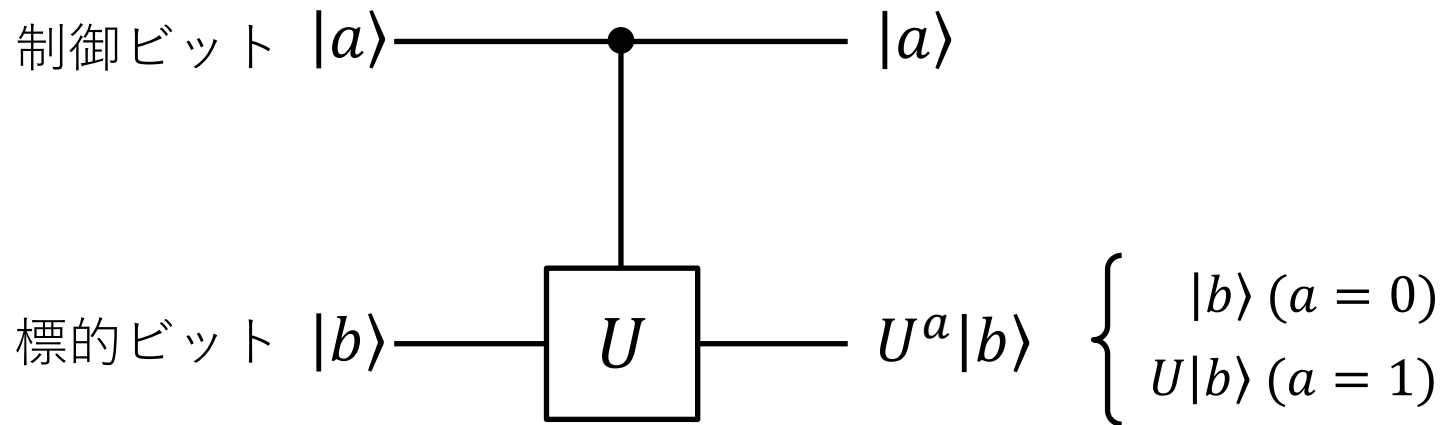
$$|x\rangle = |a_1\rangle|a_2\rangle\cdots|a_n\rangle \xrightarrow{n} \boxed{H^{\otimes n}} \longrightarrow \frac{1}{\sqrt{2^n}} \sum_y (-1)^{x \cdot y} |y\rangle$$

$$x \cdot y \equiv a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n$$

$$\begin{aligned} H^{\otimes n} |x\rangle &= \frac{1}{\sqrt{2^n}} \left(\sum_{b_1=0,1} (-1)^{a_1 \cdot b_1} |b_1\rangle \right) \cdots \left(\sum_{b_n=0,1} (-1)^{a_n \cdot b_n} |b_n\rangle \right) \\ &= \frac{1}{\sqrt{2^n}} \sum_{b_1, b_2, \dots, b_n} (-1)^{a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n} |b_1 b_2 \cdots b_n\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_y (-1)^{x \cdot y} |y\rangle \end{aligned}$$

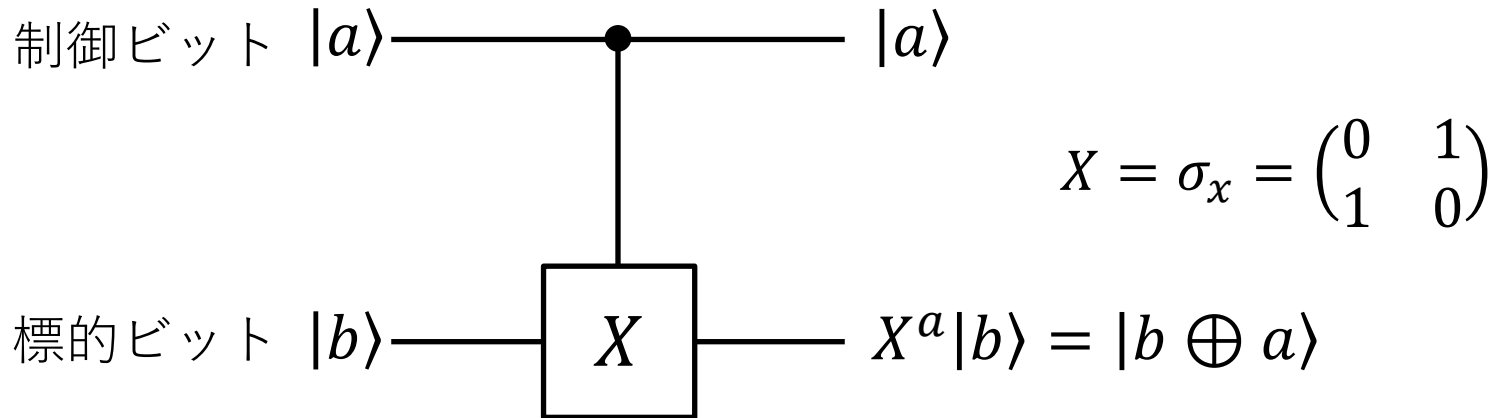
制御 U ゲート

制御ビットの状態に応じて標的ビットに U が作用



制御NOTゲート

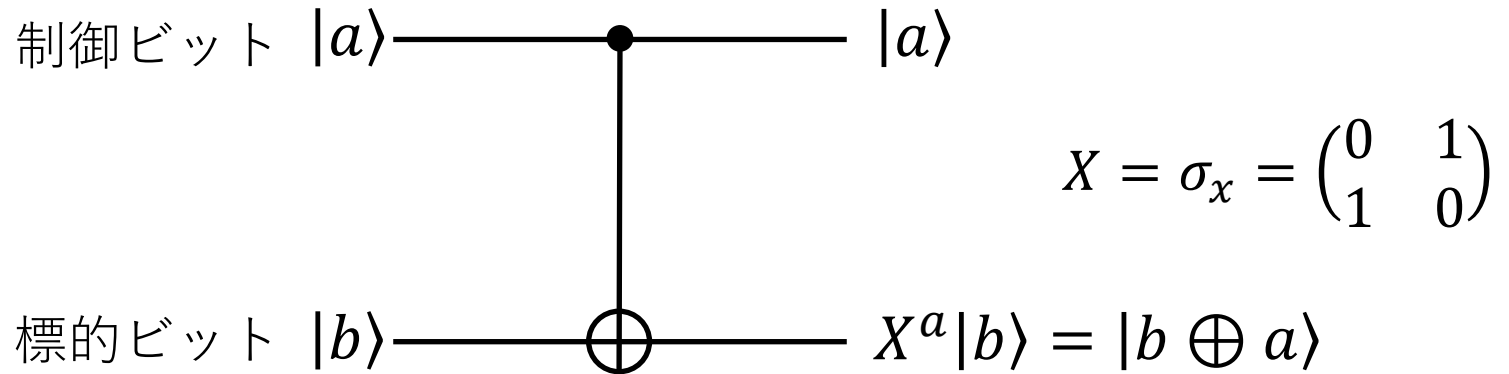
制御ビットが1のとき標的ビットを反転



$$\begin{aligned} \text{CNOT}|00\rangle &= |00\rangle \\ \text{CNOT}|01\rangle &= |01\rangle \\ \text{CNOT}|10\rangle &= |11\rangle \\ \text{CNOT}|11\rangle &= |10\rangle \end{aligned} \iff \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

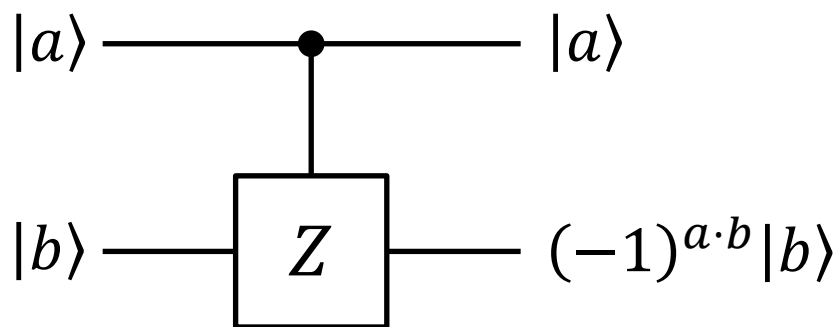
制御NOTゲート

制御ビットが1のとき標的ビットを反転



$$\begin{aligned} \text{CNOT}|00\rangle &= |00\rangle \\ \text{CNOT}|01\rangle &= |01\rangle \\ \text{CNOT}|10\rangle &= |11\rangle \\ \text{CNOT}|11\rangle &= |10\rangle \end{aligned} \iff \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

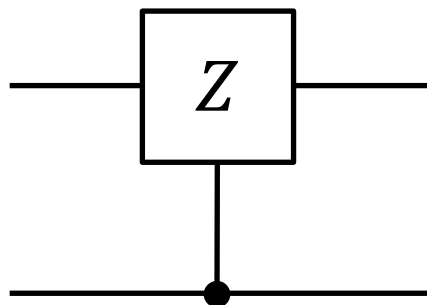
制御Zゲート



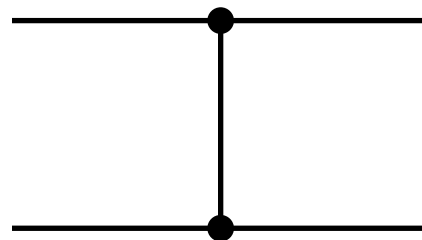
$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

||

$$|a\rangle|b\rangle \xrightarrow{CZ} (-1)^{a \cdot b} |a\rangle|b\rangle$$

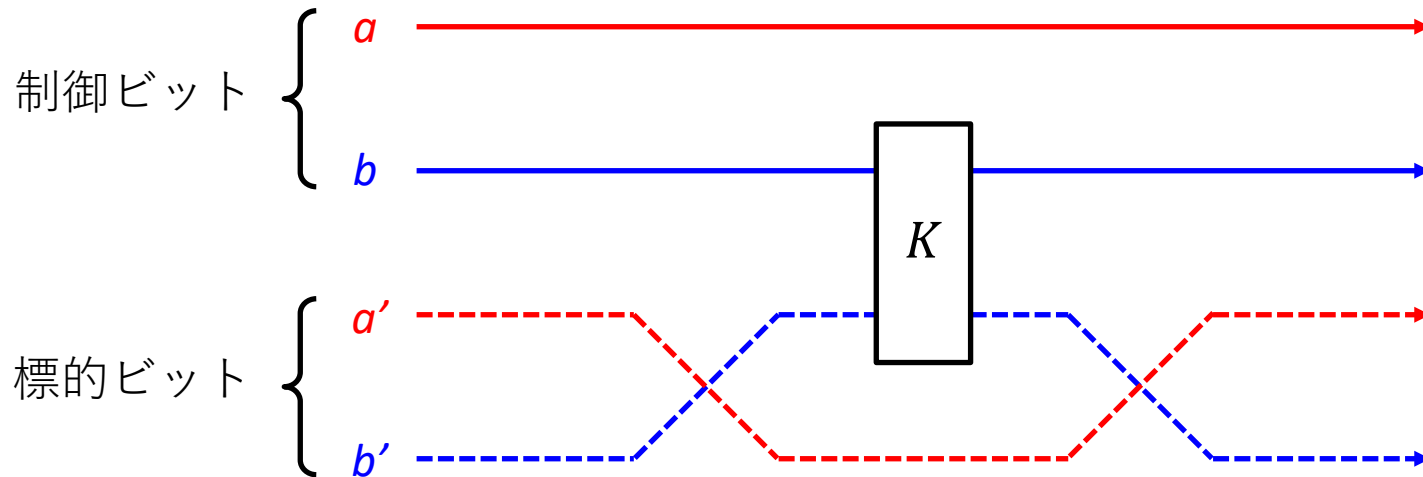


≡

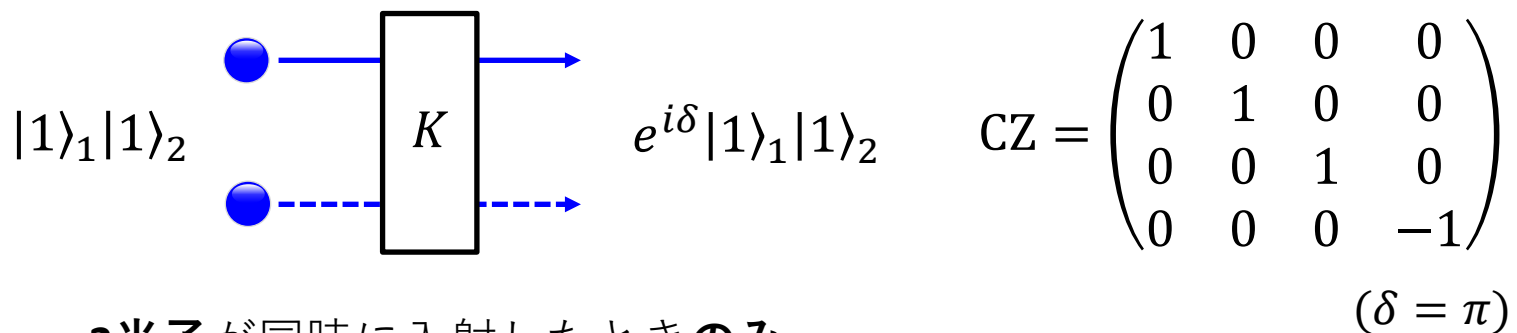


“非局所的(non-local)”

経路ビットにおける制御ゲート

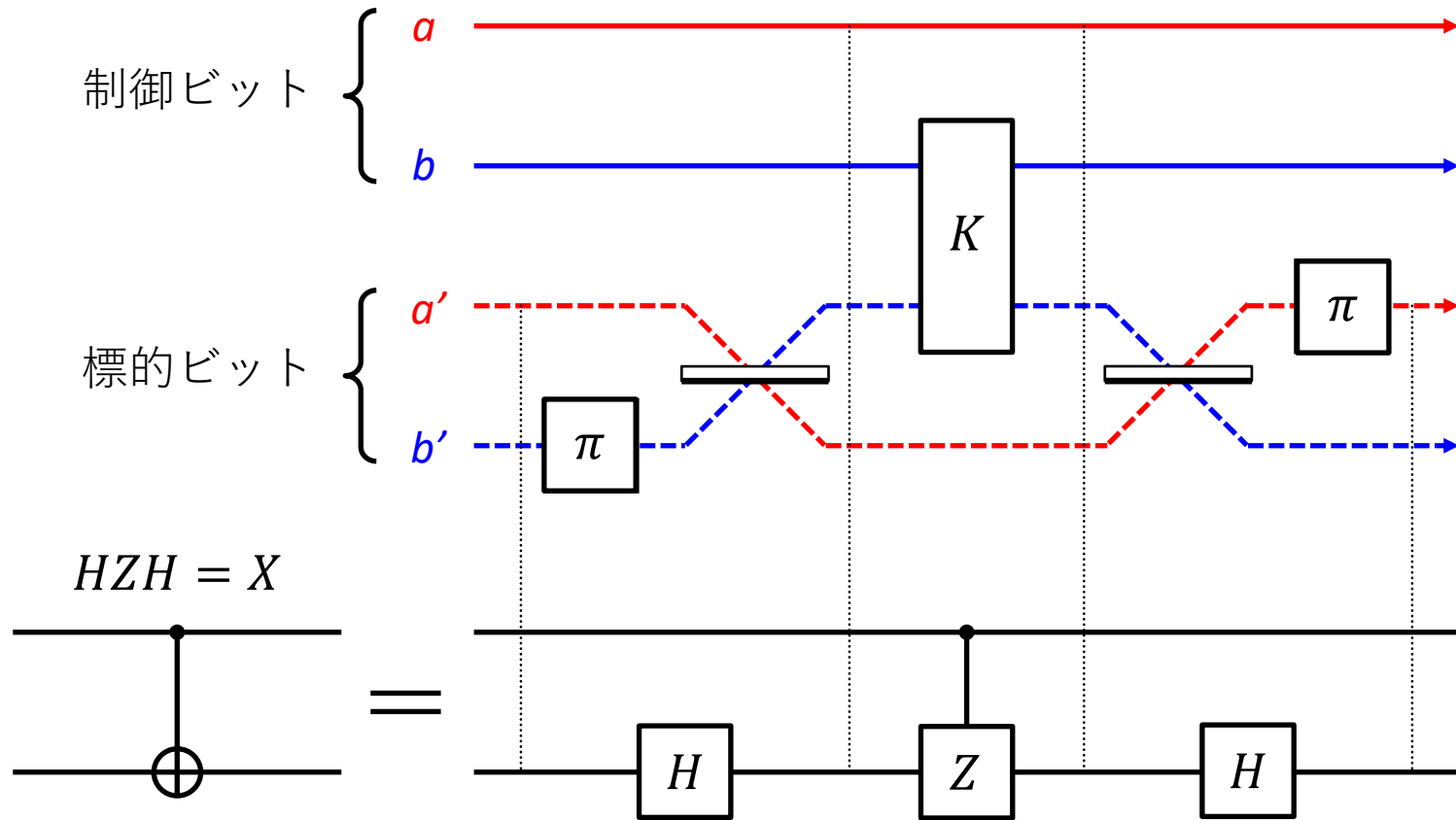


非線形カー媒体

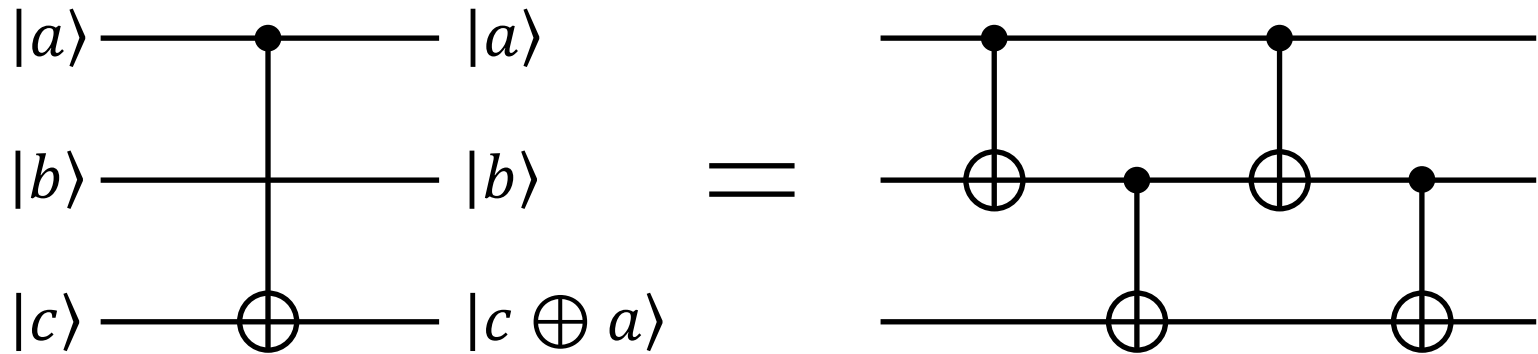


2光子が同時に入射したときのみ
位相変化 δ を与える素子(実際に作るのは大変)

経路ビットにおける制御ゲート



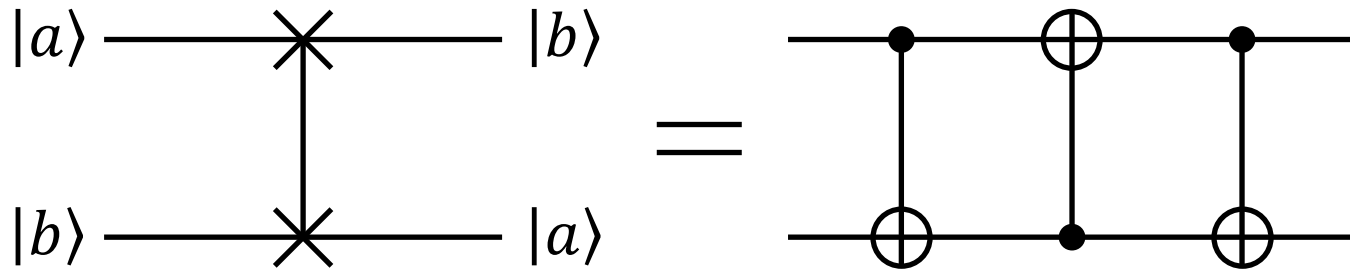
制御NOTゲート



- 第1ビットの情報を第3ビットに伝える
- 第2ビットの情報を消去する

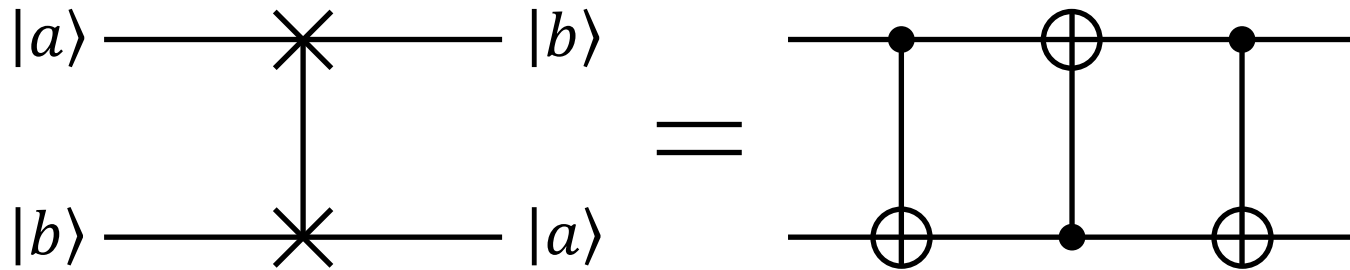
$$\begin{aligned}
 |a\rangle|b\rangle|c\rangle &\xrightarrow{C_{12}} |a\rangle|b \oplus a\rangle|c\rangle \\
 &\xrightarrow{C_{23}} |a\rangle|b \oplus a\rangle|c \oplus b \oplus a\rangle \\
 &\xrightarrow{C_{12}} |a\rangle|(b \oplus a) \oplus a\rangle|c \oplus b \oplus a\rangle = |a\rangle|b\rangle|c \oplus b \oplus a\rangle \\
 &\xrightarrow{C_{23}} |a\rangle|b\rangle|(c \oplus b \oplus a) \oplus b\rangle = |a\rangle|b\rangle|c \oplus a\rangle
 \end{aligned}$$

SWAPゲート



$$\begin{aligned} \text{SWAP}|00\rangle &= |00\rangle \\ \text{SWAP}|01\rangle &= |10\rangle \\ \text{SWAP}|10\rangle &= |01\rangle \\ \text{SWAP}|11\rangle &= |11\rangle \end{aligned} \iff \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SWAPゲート



SWAPに必要な手続き

- 第1ビットの情報を第2ビットへ書き込む
- 第2ビットの情報を第1ビットへ書き込む
- 各ビットから元の情報を消去する

$$\begin{aligned} |a\rangle|b\rangle &\xrightarrow{C_{12}} |a\rangle|b \oplus a\rangle \\ &\xrightarrow{C_{21}} |a \oplus (b \oplus a)\rangle|b \oplus a\rangle = |b\rangle|b \oplus a\rangle \\ &\xrightarrow{C_{12}} |b\rangle|(b \oplus a) \oplus b\rangle = |b\rangle|a\rangle \end{aligned}$$

ユニバーサル量子ゲート

- **1量子ビットゲートとCNOT**

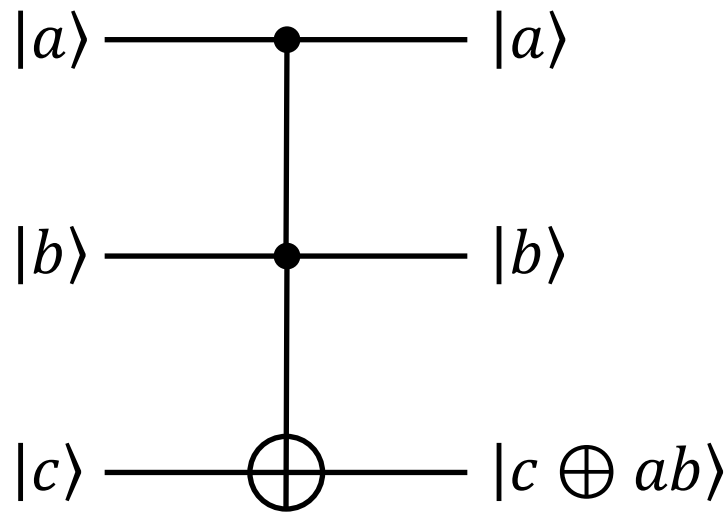
- 1量子ビットゲートとCNOTさえ実現できれば、ほかの全ての n 量子ビットゲートはその組み合わせで実行可能

- **H, S, T とCNOT**

- 離散セットのゲートでも任意の精度であらゆるゲートを近似可能

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Toffoliゲート

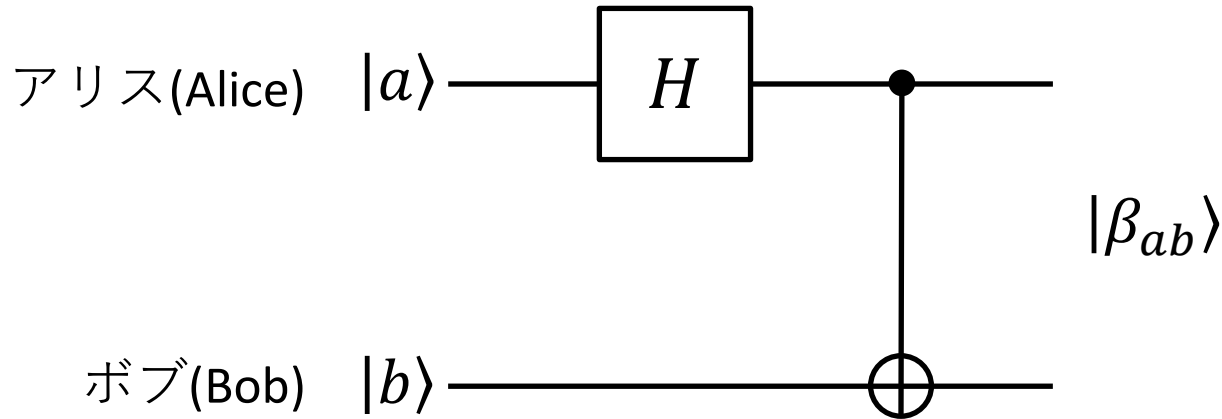


$$\text{Toffoli} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$ab = 1 \Leftrightarrow a = b = 1$$

“制御-制御NOT(C^2 -NOT)”

ベル状態



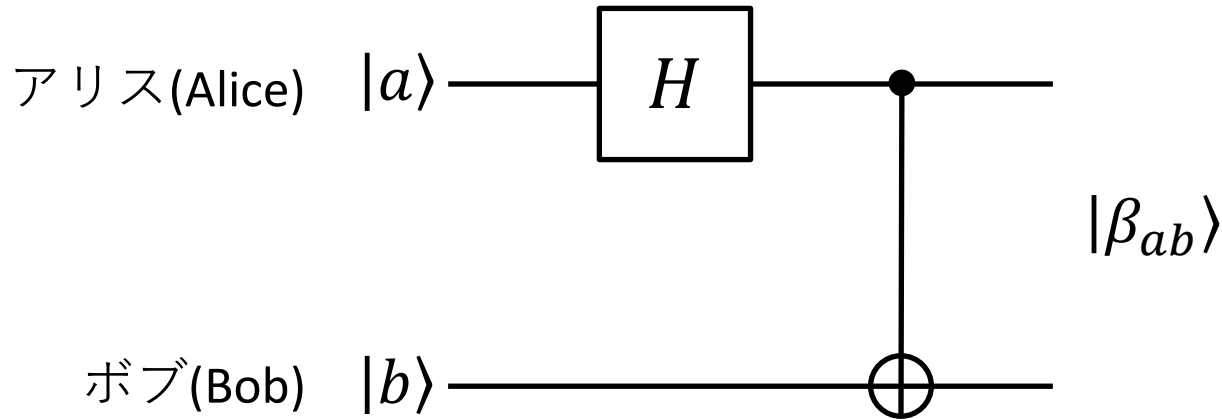
John Bell
(1928–1990)

(from Wikipedia)

$$\begin{array}{l}
 |0\rangle|0\rangle \\
 |1\rangle|0\rangle
 \end{array}
 \xrightarrow{H}
 \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)|0\rangle
 \xrightarrow{\text{CNOT}}
 \frac{1}{\sqrt{2}} (|0\rangle|0\rangle \pm |1\rangle|1\rangle)$$

$$\begin{array}{l}
 |0\rangle|1\rangle \\
 |1\rangle|1\rangle
 \end{array}
 \xrightarrow{H}
 \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)|1\rangle
 \xrightarrow{\text{CNOT}}
 \frac{1}{\sqrt{2}} (|0\rangle|1\rangle \pm |1\rangle|0\rangle)$$

ベル状態



John Bell
(1928–1990)

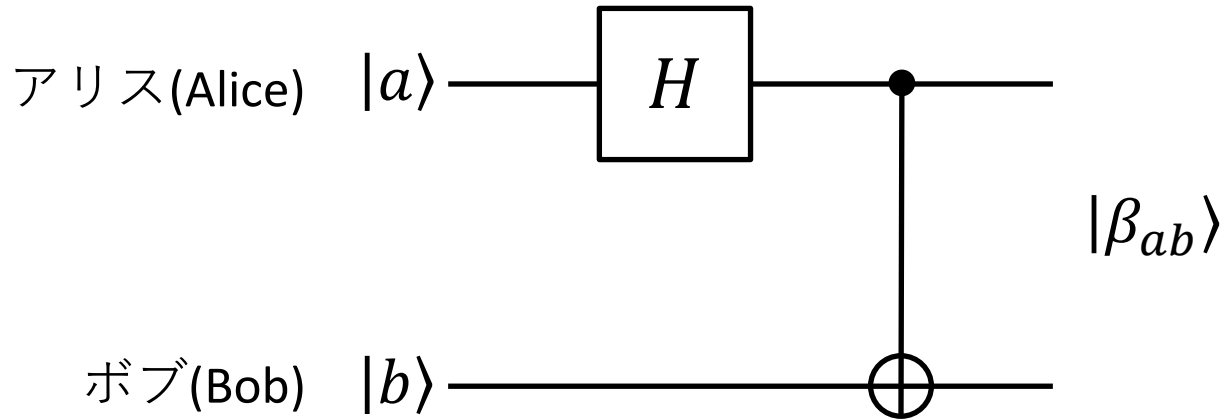
(from Wikipedia)

$$\begin{array}{l}
 \begin{array}{l} |0\rangle|0\rangle \\ |1\rangle|0\rangle \end{array} \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)|0\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|0\rangle|0\rangle \pm |1\rangle|1\rangle) \\
 \begin{array}{l} |0\rangle|1\rangle \\ |1\rangle|1\rangle \end{array} \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)|1\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|0\rangle|1\rangle \pm |1\rangle|0\rangle)
 \end{array}$$

計算基底

ベル基底

ベル状態



John Bell
(1928–1990)

(from Wikipedia)

$$|ab\rangle \xleftarrow{H} \frac{|0b\rangle + (-1)^a |1b\rangle}{\sqrt{2}} \xleftarrow{\text{CNOT}} \frac{|0b\rangle + (-1)^a |1\bar{b}\rangle}{\sqrt{2}} = |\beta_{ab}\rangle$$

$$|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

$$|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$$

$$|\beta_{10}\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$

$$|\beta_{11}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

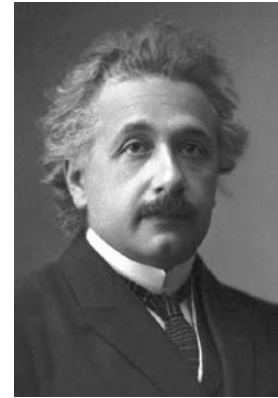
量子もつれ

ベル状態(対、ペア) (Entanglement)

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$$

またはEPR状態とも呼ばれる

→ Einstein-Podolsky-Rosen



Albert Einstein
(1879–1955)

©Nobel Foundation

MAY 15, 1935

PHYSICAL REVIEW

VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in

quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

量子もつれ

ベル状態

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$$

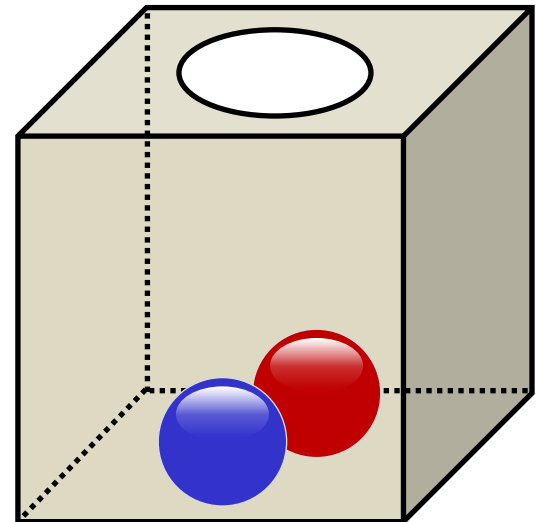
アリスが“0”を得るとボブは“1”

アリスが“1”を得るとボブは“0”

箱の中のボール...と同じこと?

アリスが“赤”を得るとボブは“青”

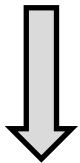
アリスが“青”を得るとボブは“赤”



量子もつれ

ベル状態

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$$



基底の変換 $\{|0\rangle, |1\rangle\} \leftrightarrow \{|\rightarrow\rangle, |\leftarrow\rangle\}$

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2^3}} [(|\rightarrow\rangle + |\leftarrow\rangle)(|\rightarrow\rangle - |\leftarrow\rangle) - (|\rightarrow\rangle - |\leftarrow\rangle)(|\rightarrow\rangle + |\leftarrow\rangle)]$$

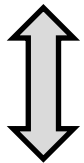
$$= \frac{1}{\sqrt{2^3}} (\cancel{|\rightarrow\rangle|\rightarrow\rangle} - \boxed{|\rightarrow\rangle|\leftarrow\rangle} + \boxed{|\leftarrow\rangle|\rightarrow\rangle} - \cancel{|\leftarrow\rangle|\leftarrow\rangle} \\ - \cancel{|\rightarrow\rangle|\rightarrow\rangle} - \boxed{|\rightarrow\rangle|\leftarrow\rangle} + \boxed{|\leftarrow\rangle|\rightarrow\rangle} + \cancel{|\leftarrow\rangle|\leftarrow\rangle})$$

$$= \frac{1}{\sqrt{2}} (|\leftarrow\rangle_A |\rightarrow\rangle_B - |\rightarrow\rangle_A |\leftarrow\rangle_B)$$

量子もつれ

ベル状態

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$$



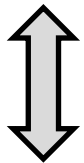
$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|\leftarrow\rangle_A |\rightarrow\rangle_B - |\rightarrow\rangle_A |\leftarrow\rangle_B)$$

- もつれた状態は基底を変えても、もつれている
- アリスとボブが異なる基底で測定すると相関はない
→ アリスが計算(Z)基底で“0”を得たのち、ボブがX基底で測定を行うと“→”も“←”も50%の確率で得られる

量子もつれ

ベル状態

$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$$



$$|\beta_{11}\rangle_{AB} = \frac{1}{\sqrt{2}} (|\leftarrow\rangle_A |\rightarrow\rangle_B - |\rightarrow\rangle_A |\leftarrow\rangle_B)$$

レポート課題・第2問(8点)

任意の基底 $\{|\psi\rangle, |\psi'\rangle\}$ で $|\beta_{11}\rangle_{AB}$ がもつれていることを確認せよ。



レポート課題・第1問で求めた

講義内容

- スピンと量子ビット
- 量子力学の公理と量子ゲート
- **量子テレポーテーション**

量子テレポーテーション

Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels

Charles H. Bennett,⁽¹⁾ Gilles Brassard,⁽²⁾ Claude Crépeau,^{(2),(3)}
Richard Jozsa,⁽²⁾ Asher Peres,⁽⁴⁾ and William K. Wootters⁽⁵⁾

⁽¹⁾ *IBM Research Division, T.J. Watson Research Center, Yorktown Heights, New York 10598*

⁽²⁾ *Département IRO, Université de Montréal, C.P. 6128, Succursale "A", Montréal, Québec, Canada H3C 3J7*

⁽³⁾ *Laboratoire d'Informatique de l'École Normale Supérieure, 45 rue d'Ulm, 75230 Paris CEDEX 05, France^(a)*

⁽⁴⁾ *Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel*

⁽⁵⁾ *Department of Physics, Williams College, Williamstown, Massachusetts 01267*

(Received 2 December 1992)

An unknown quantum state $|\phi\rangle$ can be disassembled into, then later reconstructed from, purely classical information and purely nonclassical Einstein-Podolsky-Rosen (EPR) correlations. To do so the sender, "Alice," and the receiver, "Bob," must prearrange the sharing of an EPR-correlated pair of particles. Alice makes a joint measurement on her EPR particle and the unknown quantum system, and sends Bob the classical result of this measurement. Knowing this, Bob can convert the state of his EPR particle into an exact replica of the unknown state $|\phi\rangle$ which Alice destroyed.



Charles Bennett
(1943–)

©Aya Furuta

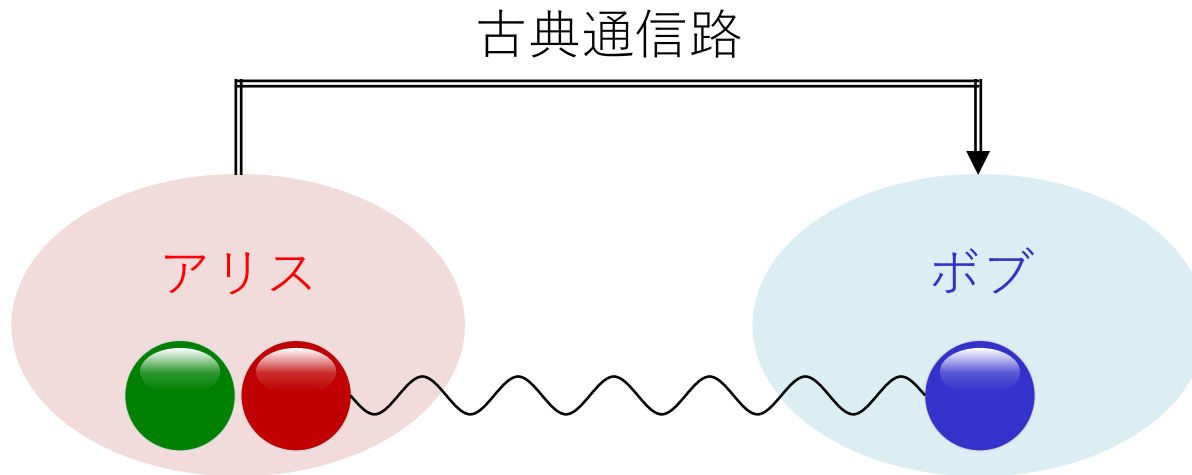
目的: 未知の量子状態を、状態を壊す(読む)ことなく、
古典通信路と量子もつれ対を用いて遠隔地に再生する

Phys. Rev. Lett. **70**, 1895 (1993) Bennett *et al.*

量子テレポーテーション

設定(必要な道具立て)

- ✓ $|\psi\rangle$ の内容はアリスも知らない
- ✓ アリスとボブが量子もつれ状態を事前に共有している
- ✓ アリスとボブの間に古典通信路が確保されている
- ✓ アリスは2量子ビットのベル測定を行うことができる
- ✓ ボブは1量子ビットゲートを行うことができる



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad |\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

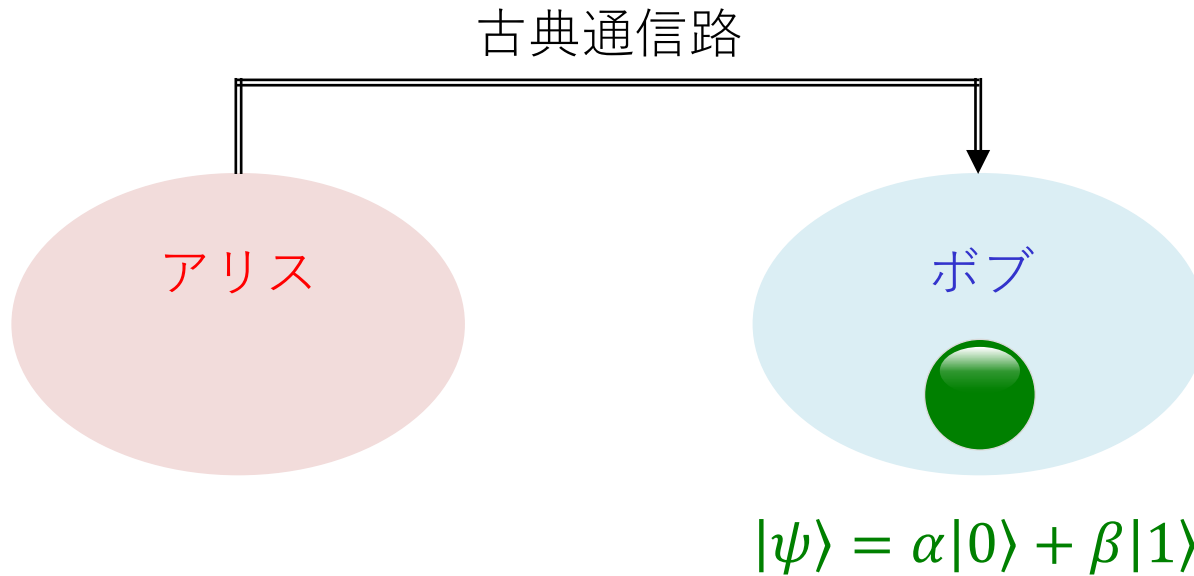
未知の状態

もつれ対

量子テレポーテーション

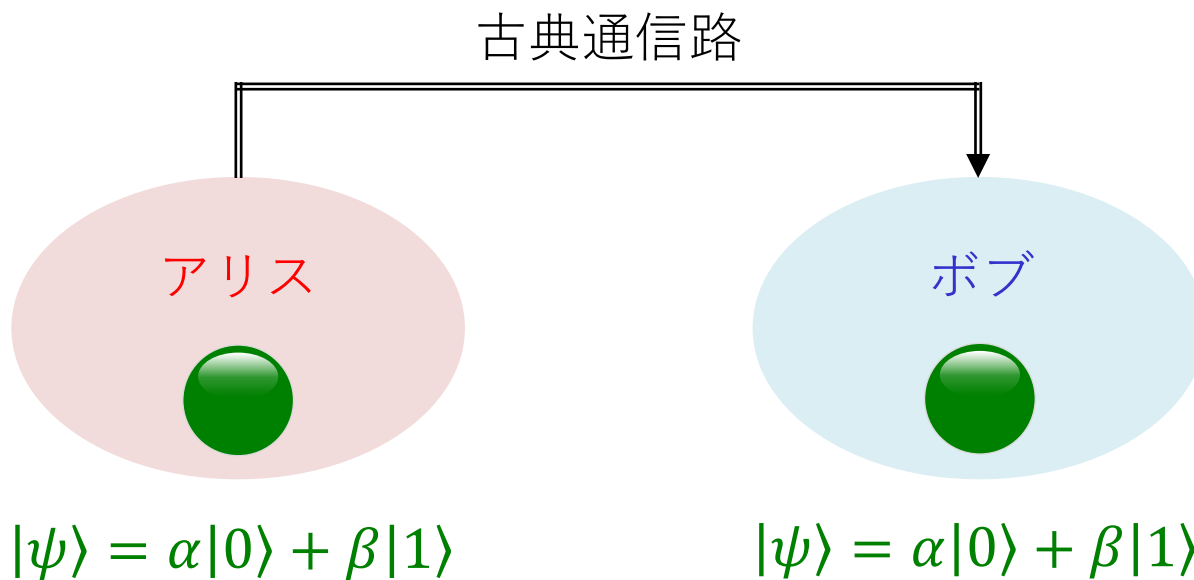
設定(必要な道具立て)

- ✓ $|\psi\rangle$ の内容はアリスも知らない
- ✓ アリスとボブが量子もつれ状態を事前に共有している
- ✓ アリスとボブの間に古典通信路が確保されている
- ✓ アリスは2量子ビットのベル測定を行うことができる
- ✓ ボブは1量子ビットゲートを行うことができる



量子FAX?

アリスが状態を保持したまま、ボブが同じ状態を生成することはできるか?



複製禁止定理

任意の状態 $|\psi\rangle$ に対して $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ となる
ユニタリ演算子 U は存在しない

LETTERS TO NATURE

A single quantum cannot be cloned

W. K. Wootters*

Center for Theoretical Physics, The University of Texas at Austin,
Austin, Texas 78712, USA

W. H. Zurek

Theoretical Astrophysics 130-33, California Institute of Technology,
Pasadena, California 91125, USA

If a photon of definite polarization encounters an excited atom, there is typically some nonvanishing probability that the atom will emit a second photon by stimulated emission. Such a photon is guaranteed to have the same polarization as the original photon. But is it possible by this or any other process to amplify a quantum state, that is, to produce several copies of a quantum system (the polarized photon in the present case) each having the same state as the original? If it were, the amplifying process could be used to ascertain the exact state of a quantum system: in the case of a photon, one could determine its polarization by first producing a beam of identically polarized copies and then measuring the Stokes parameters¹. We show here that the linearity of quantum mechanics forbids such replication and that this conclusion holds for all quantum systems.

Note that if photons could be cloned, a plausible argument could be made for the possibility of faster-than-light communication². It is well known that for certain non-separably correlated Einstein-Podolsky-Rosen pairs of photons, once an observer has made a polarization measurement (say, vertical versus horizontal) on one member of the pair, the other one, which may be far away, can be for all purposes of prediction regarded as having the same polarization³. If this second photon could be replicated and its precise polarization measured as above, it would be possible to ascertain whether, for example, the first photon had been subjected to a measurement of linear or circular polarization. In this way the first observer would be able to transmit information faster than light by encoding his message into his choice of measurement. The actual impossibility of cloning photons, shown below, thus prohibits superluminal communication by this scheme. That such a scheme must fail for some reason despite the well-established existence of long-range quantum correlations^{4,8}, is a general consequence of quantum mechanics⁹.

A perfect amplifying device would have the following effect

on an incoming photon with polarization state $|s\rangle$:

$$|A_0\rangle|s\rangle \rightarrow |A_s\rangle|ss\rangle \quad (1)$$

Here $|A_0\rangle$ is the 'ready' state of the apparatus, and $|A_s\rangle$ is its final state, which may or may not depend on the polarization of the original photon. The symbol $|ss\rangle$ refers to the state of the radiation field in which there are two photons each having the polarization $|s\rangle$. Let us suppose that such an amplification can in fact be accomplished for the vertical polarization $|\uparrow\rangle$ and for the horizontal polarization $|\leftrightarrow\rangle$. That is,

$$|A_0\rangle|\uparrow\rangle \rightarrow |A_{\text{vert}}\rangle|\uparrow\uparrow\rangle \quad (2)$$

and

$$|A_0\rangle|\leftrightarrow\rangle \rightarrow |A_{\text{hor}}\rangle|\leftrightarrow\leftrightarrow\rangle \quad (3)$$

According to quantum mechanics this transformation should be representable by a linear (in fact unitary) operator. It therefore follows that if the incoming photon has the polarization given by the linear combination $\alpha|\uparrow\rangle + \beta|\leftrightarrow\rangle$ —for example, it could be linearly polarized in a direction 45° from the vertical, so that $\alpha = \beta = 2^{-1/2}$ —the result of its interaction with the apparatus will be the superposition of equations (2) and (3):

$$|A_0\rangle(\alpha|\uparrow\rangle + \beta|\leftrightarrow\rangle) \rightarrow \alpha|A_{\text{vert}}\rangle|\uparrow\uparrow\rangle + \beta|A_{\text{hor}}\rangle|\leftrightarrow\leftrightarrow\rangle \quad (4)$$

If the apparatus states $|A_{\text{vert}}\rangle$ and $|A_{\text{hor}}\rangle$ are not identical, then the two photons emerging from the apparatus are in a mixed state of polarization. If these apparatus states are identical, then the two photons are in the pure state

$$\alpha|\uparrow\uparrow\rangle + \beta|\leftrightarrow\leftrightarrow\rangle \quad (5)$$

In neither of these cases is the final state the same as the state with two photons both having the polarization $\alpha|\uparrow\rangle + \beta|\leftrightarrow\rangle$. That state, the one which would be required if the apparatus were to be a perfect amplifier, can be written as

$$2^{-1/2}(\alpha a_{\text{vert}}^+ + \beta a_{\text{hor}}^+)^2|0\rangle = \alpha^2|\uparrow\uparrow\rangle + 2^{1/2}\alpha\beta|\uparrow\leftrightarrow\rangle + \beta^2|\leftrightarrow\leftrightarrow\rangle$$

which is a pure state different from the one obtained above by superposition [equation (5)].

Thus no apparatus exists which will amplify an arbitrary polarization. The above argument does not rule out the possibility of a device which can amplify two special polarizations, such as vertical and horizontal. Indeed, any measuring device which distinguishes between these two polarizations, a Nicol prism for example, could be used to trigger such an amplification.

The same argument can be applied to any other kind of quantum system. As in the case of photons, linearity does not forbid the amplification of any given state by a device designed especially for that state, but it does rule out the existence of a device capable of amplifying an arbitrary state.

Nature Vol. 299 28 October 1982

803

Milonni (unpublished work) has shown that the process of stimulated emission does not lead to quantum amplification, because if there is stimulated emission there must also be—with equal probability in the case of one incoming photon—spontaneous emission, and the polarization of a spontaneously emitted photon is entirely independent of the polarization of the original.

It is conceivable that a more sophisticated amplifying apparatus could get around Milonni's argument. We have therefore presented the above simple argument, based on the linearity of quantum mechanics, to show that no apparatus, however complicated, can amplify an arbitrary polarization.

We stress that the question of replicating individual photons is of practical interest. It is obviously closely related to the

quantum limits on the noise in amplifiers^{10,11}. Moreover, an experiment devised to establish the extent to which polarization of single photons can be replicated through the process of stimulated emission is under way (A. Gozzini, personal communication; and see ref. 12). The quantum mechanical prediction is quite definite; for each perfect clone there is also one randomly polarized, spontaneously emitted, photon.

We thank Alain Aspect, Carl Caves, Ron Dickman, Ted Jacobson, Peter Milonni, Marlan Scully, Pierre Meystre, Don Page and John Archibald Wheeler for enjoyable and stimulating discussions.

This work was supported in part by the NSF (PHY 78-26592 and AST 79-22012-A1). W.H.Z. acknowledges a Richard Chace Tolman Fellowship.

Received 11 August; accepted 7 September 1982.

1. Born, M. & Wolf, E. *Principles of Optics* 4th edn (Pergamon, New York, 1970).
2. Herbert, N. *Found. Phys.* (in the press).
3. Einstein, A., Podolsky, B. & Rosen, N. *Phys. Rev.* **47**, 777-780 (1935).
4. Bohm, D. *Quantum Theory*, 611-623 (Prentice-Hall, Englewood Cliffs, 1951).
5. Kocher, C. A. & Commins, E. D. *Phys. Rev. Lett.* **18**, 575-578 (1967).
6. Freedman, S. J. & Clauser, J. R. *Phys. Rev. Lett.* **28**, 938-941 (1972).

7. Fry, E. S. & Thompson, R. C. *Phys. Rev. Lett.* **37**, 465-468 (1976).
8. Aspect, A., Grangier, P. & Roger, G. *Phys. Rev. Lett.* **47**, 460-463 (1981).
9. Bussey, P. J. *Phys. Lett.* **90A**, 9-12 (1982).
10. Hsu, H. A. & Mallen, J. A. *Phys. Rev.* **128**, 2407-2410 (1962).
11. Caves, C. M. *Phys. Rev.* **D15**, (in the press).
12. Gozzini, A. *Proc. Symp. on Wave-Particle Duality* (eds Diner, S., Fargue, D., Lochak, G. & Soleri, F.) (Reidel, Dordrecht, in the press).

*Present address: Department of Physics and Astronomy, Williams College, Williamstown, Massachusetts 01267, USA.

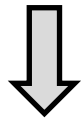
複製禁止定理

任意の状態 $|\psi\rangle$ に対して $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ となるユニタリ演算子 U は存在しない

証明1: 存在するならば...

$$U|0\rangle|0\rangle = |0\rangle|0\rangle$$

$$U|1\rangle|0\rangle = |1\rangle|1\rangle$$



$$\begin{aligned} U(\alpha|0\rangle + \beta|1\rangle)|0\rangle &= \alpha U|0\rangle|0\rangle + \beta U|1\rangle|0\rangle \\ &= \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle \\ &\neq (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \end{aligned}$$

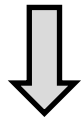
複製禁止定理

任意の状態 $|\psi\rangle$ に対して $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$ となるユニタリ演算子 U は存在しない

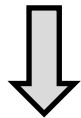
証明2: 存在するならば...

$$U|\varphi\rangle|0\rangle = |\varphi\rangle|\varphi\rangle$$

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$



$$\langle\varphi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \langle\varphi|\psi\rangle\langle\varphi|\psi\rangle$$



$$\langle\varphi|\psi\rangle = (\langle\varphi|\psi\rangle)^2$$

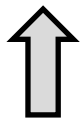
自分自身か直交状態

$$\therefore \langle\varphi|\psi\rangle = 0, 1$$

量子テレポーテーション

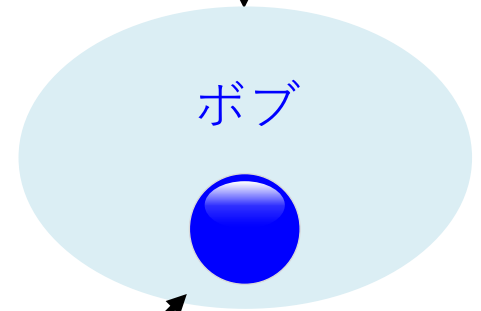
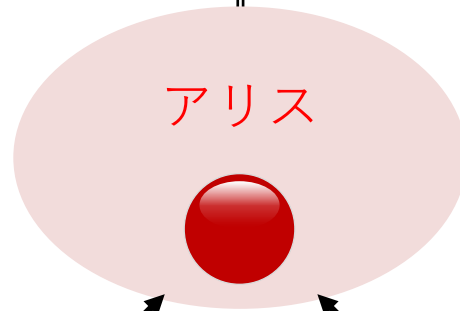
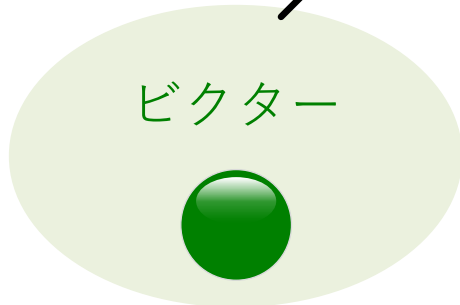
ここでの設定(本質的には同じ)

- ✓ ビクターによる $|\psi\rangle$ の送信
- ✓ もつれ源

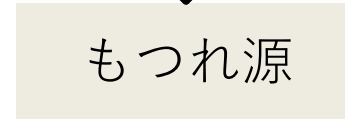
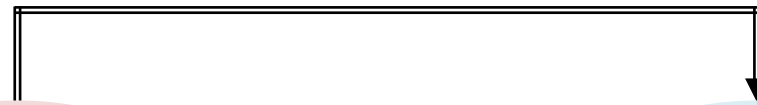


役割分担の明確化

$$|\psi\rangle_V = \alpha|0\rangle_V + \beta|1\rangle_V$$



古典通信路



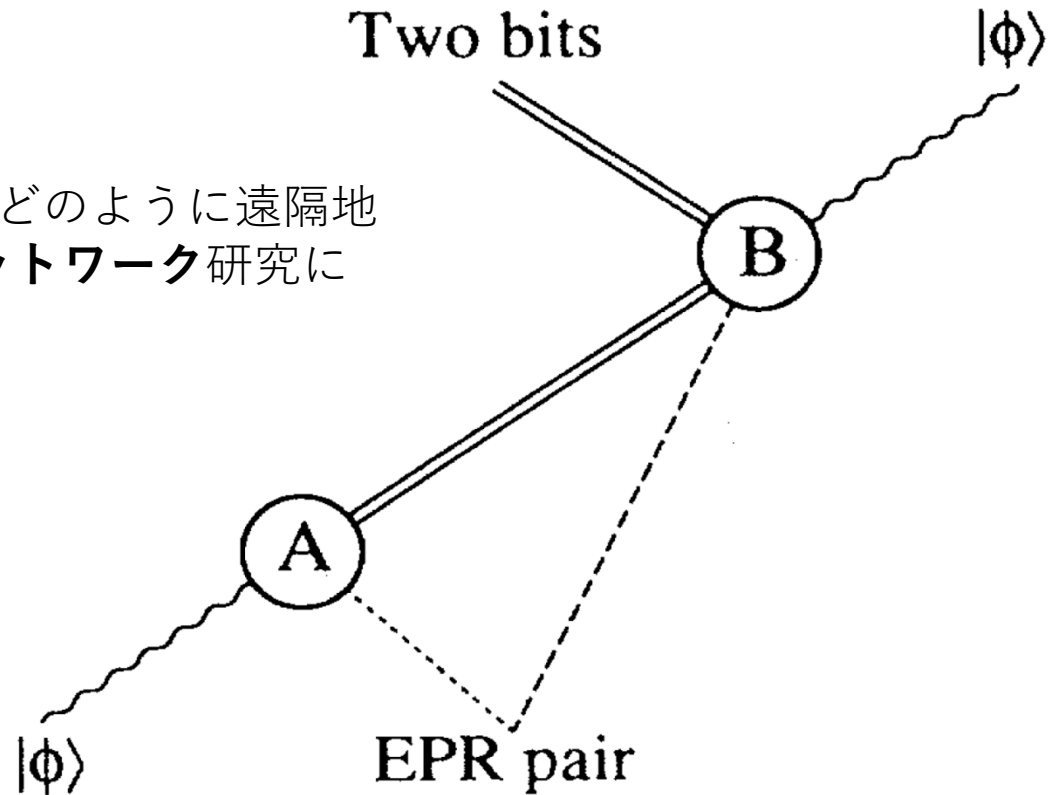
$$|\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

量子テレポーテーション

ここでの設定(本質的には同じ)

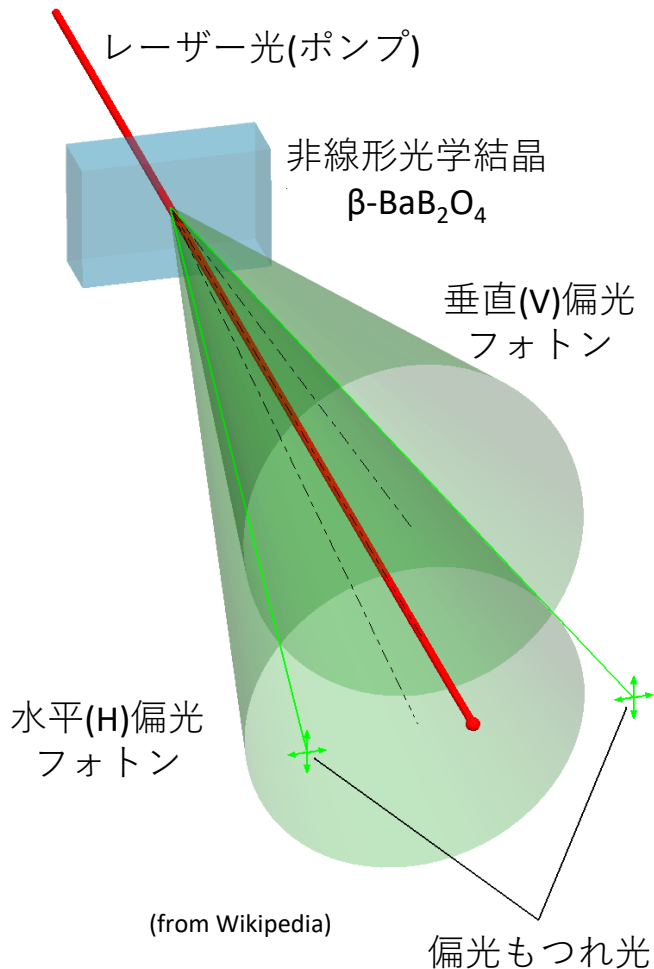
- ✓ ビクターによる $|\psi\rangle$ の送信
- ✓ もつれ源

(既知の)量子もつれ状態をどのように遠隔地に配信するかは、**量子ネットワーク**研究における主要テーマ課題



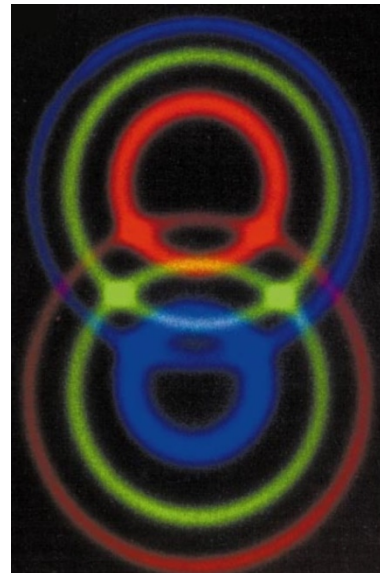
もつれ源

タイプII型パラメトリック下方変換



ポンプ1光子がエネルギー保存則、運動量保存則を満たしながら2光子に分離(頻度 $\approx 10^{-6}$)

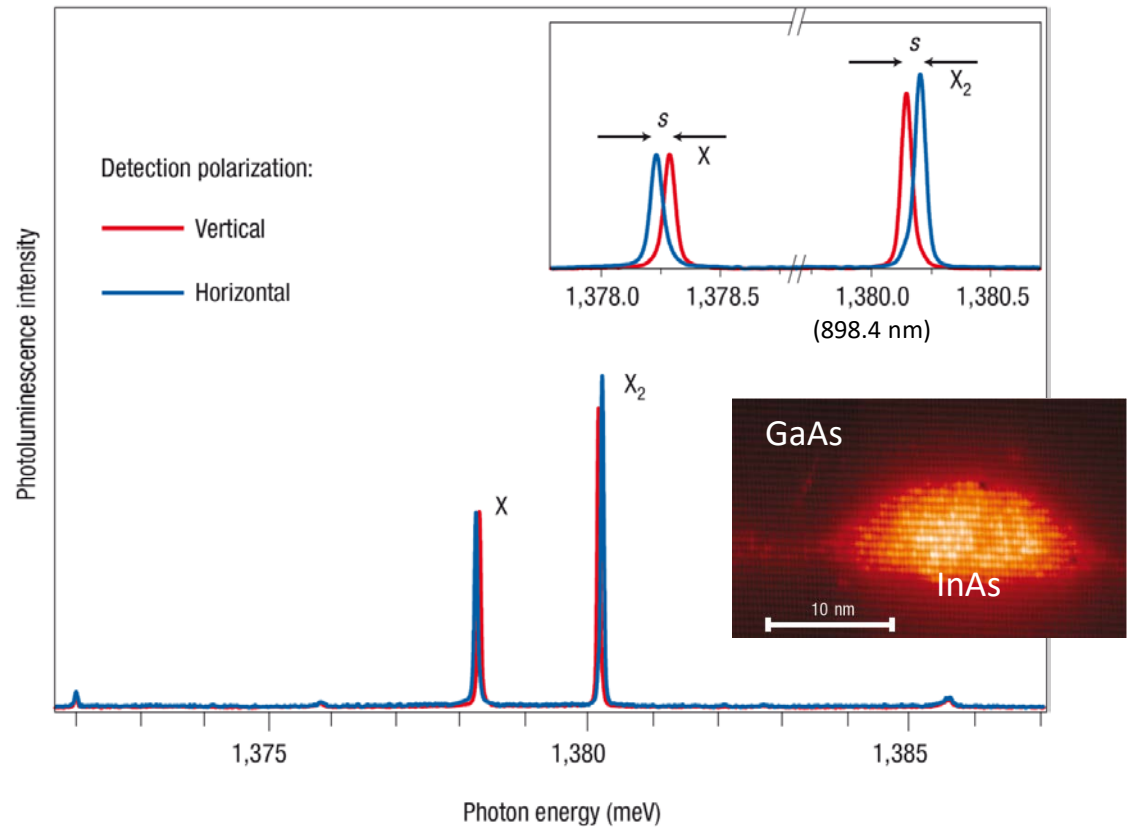
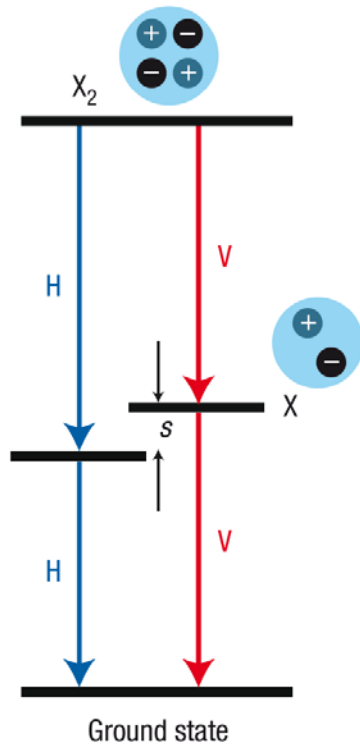
2光子の偏光が異なるケースをタイプII型と呼び、等エネルギー円錐が交差する場所ではもつれ光が生成されている



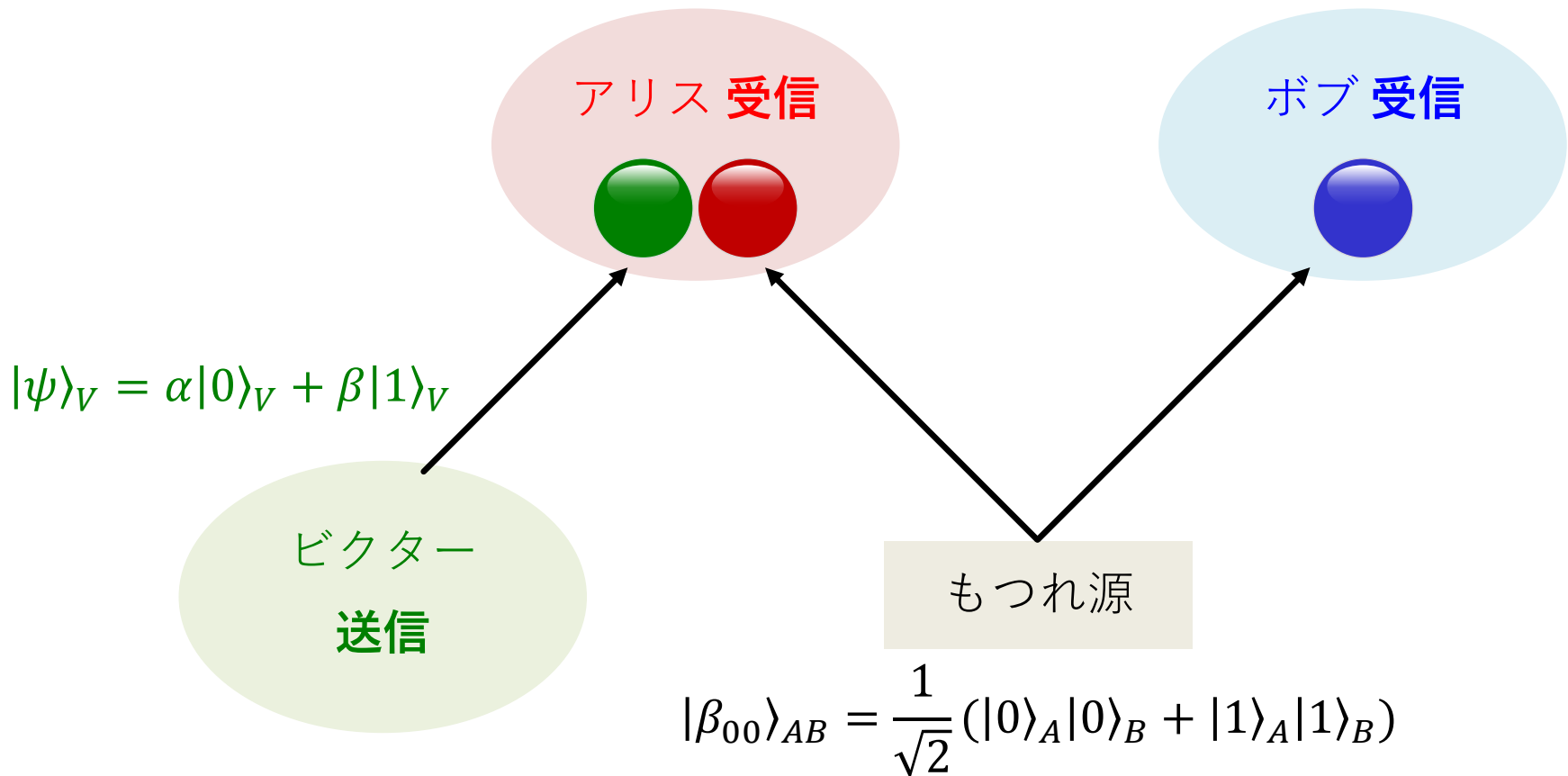
$$\frac{1}{\sqrt{2}} (|H\rangle|V\rangle + |V\rangle|H\rangle)$$

もつれ源

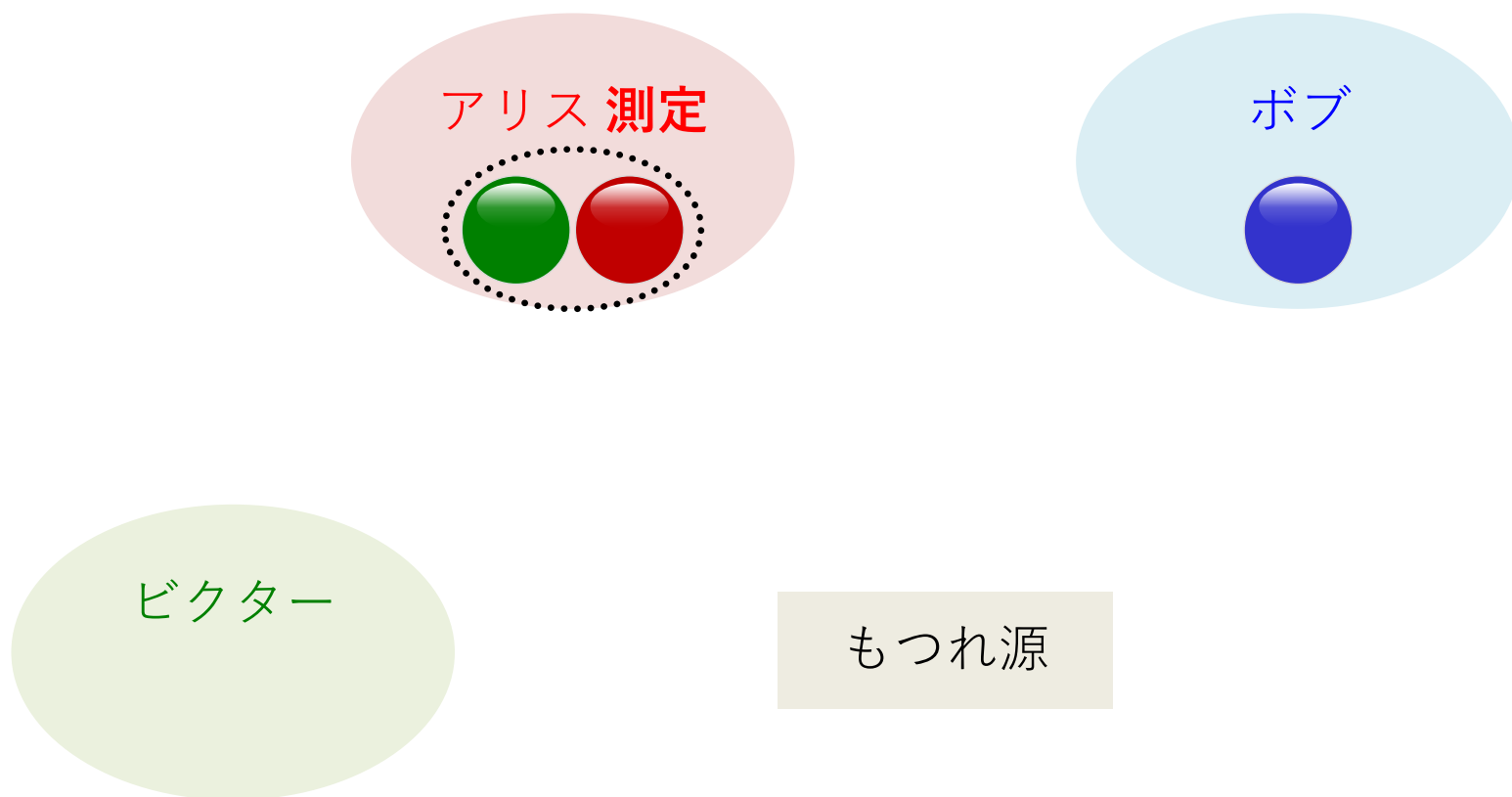
自己形成量子ドット中の励起子分子のカスケード発光



量子テレポーテーション



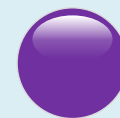
量子テレポーテーション



量子テレポーテーション

アリス 測定

ボブ

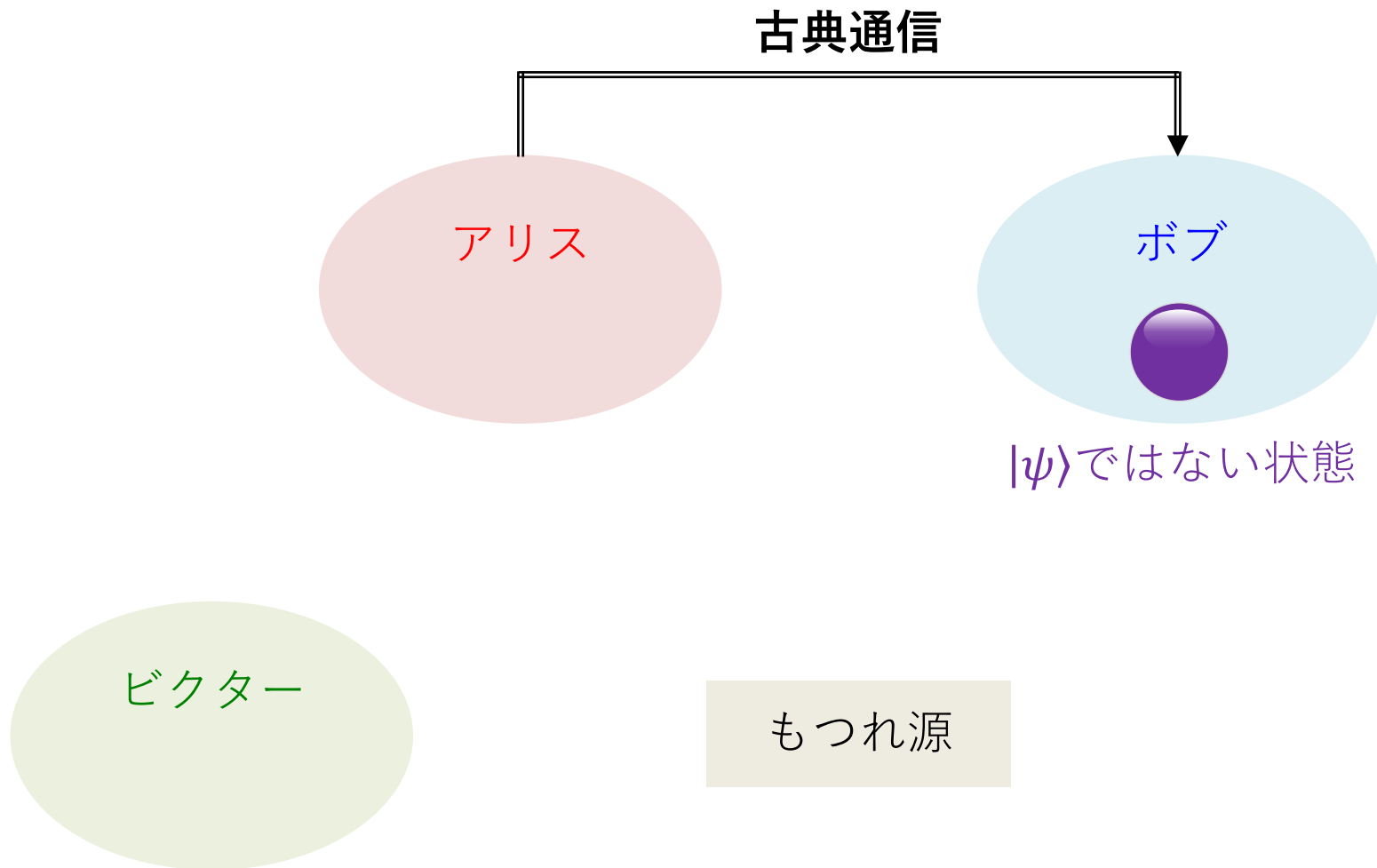


$|\psi\rangle$ ではない状態

ビクター

もつれ源

量子テレポーテーション



量子テレポーテーション

アリス

ボブ 復元



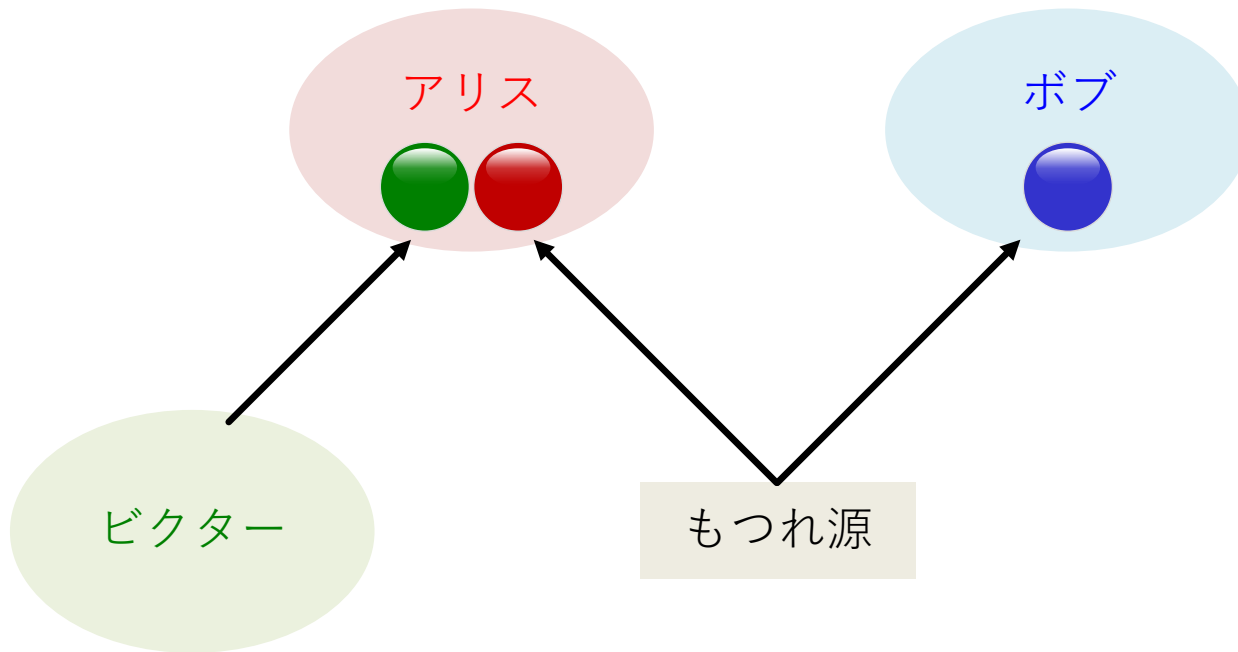
$$|\psi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B$$

ビクター

もつれ源

ステップ1: 送信

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X|\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z|\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ|\psi\rangle_B$$



$$\begin{aligned} |\psi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ Z|\psi\rangle &= \alpha|0\rangle - \beta|1\rangle \\ X|\psi\rangle &= \alpha|1\rangle + \beta|0\rangle \\ XZ|\psi\rangle &= \alpha|1\rangle - \beta|0\rangle \end{aligned}$$

$$\begin{aligned} |\beta_{00}\rangle &= (|00\rangle + |11\rangle)/\sqrt{2} \\ |\beta_{01}\rangle &= (|01\rangle + |10\rangle)/\sqrt{2} \\ |\beta_{10}\rangle &= (|00\rangle - |11\rangle)/\sqrt{2} \\ |\beta_{11}\rangle &= (|01\rangle - |10\rangle)/\sqrt{2} \end{aligned}$$

確認

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B$$

左辺 $(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{\alpha}{\sqrt{2}} |000\rangle + \frac{\beta}{\sqrt{2}} |011\rangle + \frac{\alpha}{\sqrt{2}} |100\rangle + \frac{\beta}{\sqrt{2}} |111\rangle$

右辺各項

$$|\beta_{00}\rangle |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) = \frac{\alpha}{\sqrt{2}} |000\rangle + \frac{\beta}{\sqrt{2}} |001\rangle + \frac{\alpha}{\sqrt{2}} |110\rangle + \frac{\beta}{\sqrt{2}} |111\rangle$$

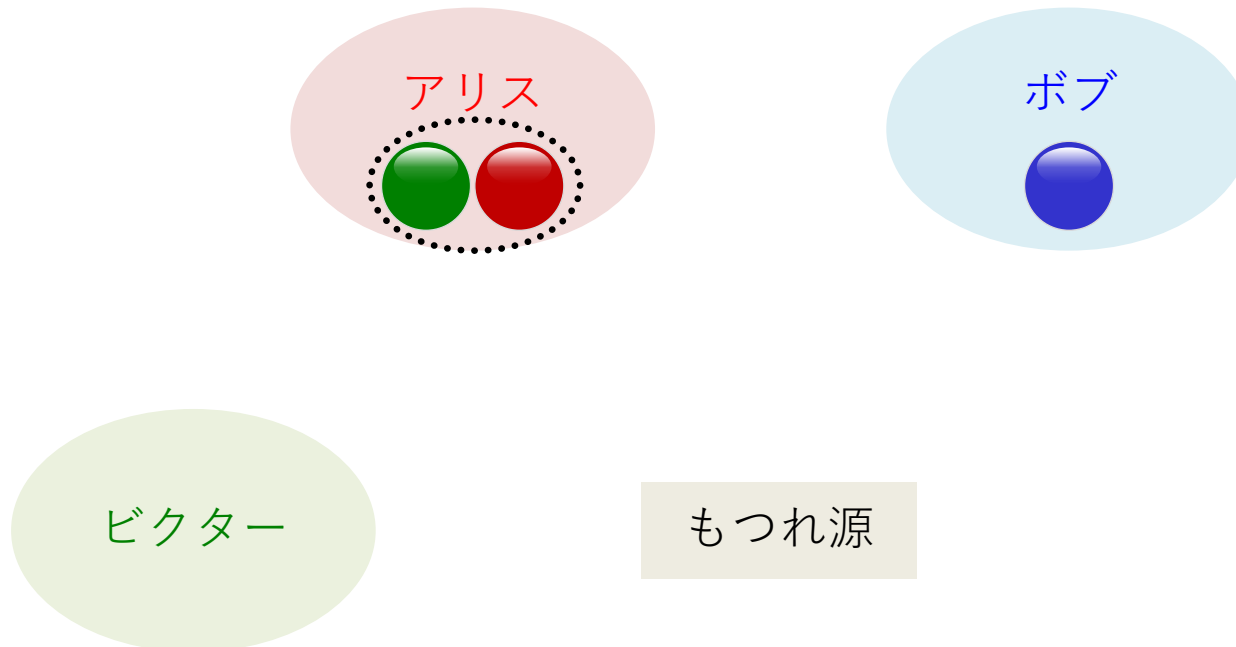
$$|\beta_{10}\rangle Z |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \otimes (\alpha|0\rangle - \beta|1\rangle) = \frac{\alpha}{\sqrt{2}} |000\rangle - \frac{\beta}{\sqrt{2}} |001\rangle - \frac{\alpha}{\sqrt{2}} |110\rangle + \frac{\beta}{\sqrt{2}} |111\rangle$$

$$|\beta_{01}\rangle X |\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \otimes (\alpha|1\rangle + \beta|0\rangle) = \frac{\alpha}{\sqrt{2}} |011\rangle + \frac{\beta}{\sqrt{2}} |010\rangle + \frac{\alpha}{\sqrt{2}} |101\rangle + \frac{\beta}{\sqrt{2}} |100\rangle$$

$$|\beta_{01}\rangle XZ |\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \otimes (\alpha|1\rangle - \beta|0\rangle) = \frac{\alpha}{\sqrt{2}} |011\rangle - \frac{\beta}{\sqrt{2}} |010\rangle - \frac{\alpha}{\sqrt{2}} |101\rangle + \frac{\beta}{\sqrt{2}} |100\rangle$$

ステップ2: ベル測定

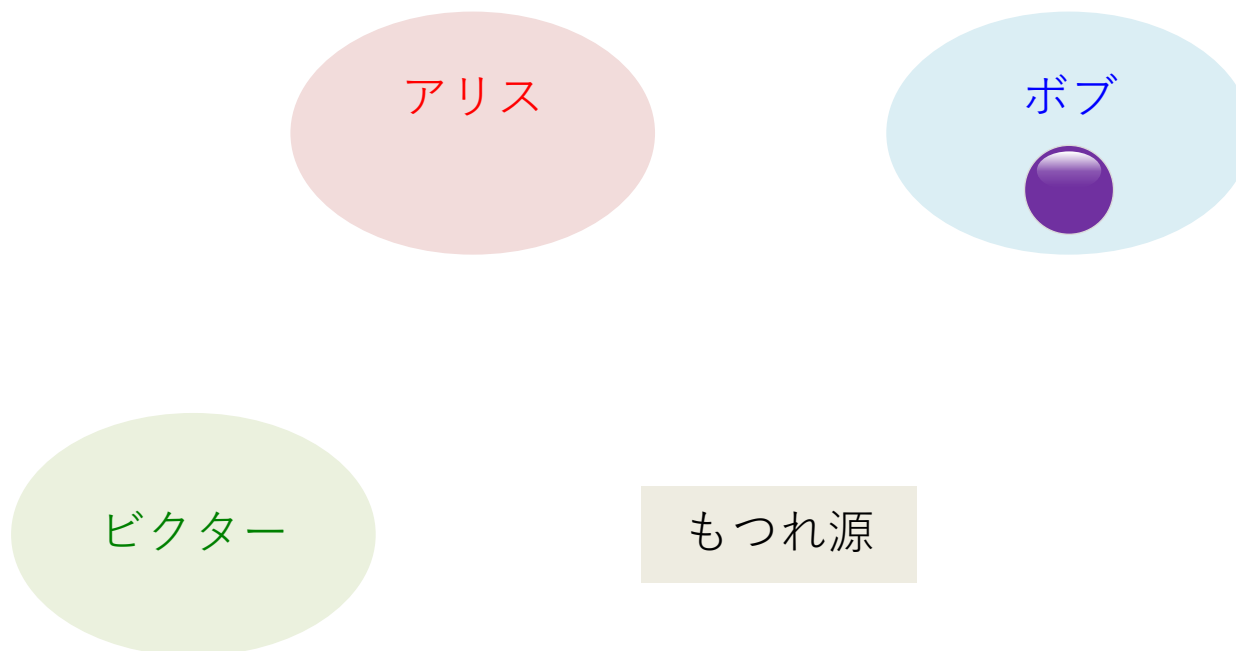
$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B$$



ステップ2: ベル測定

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X|\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z|\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ|\psi\rangle_B$$

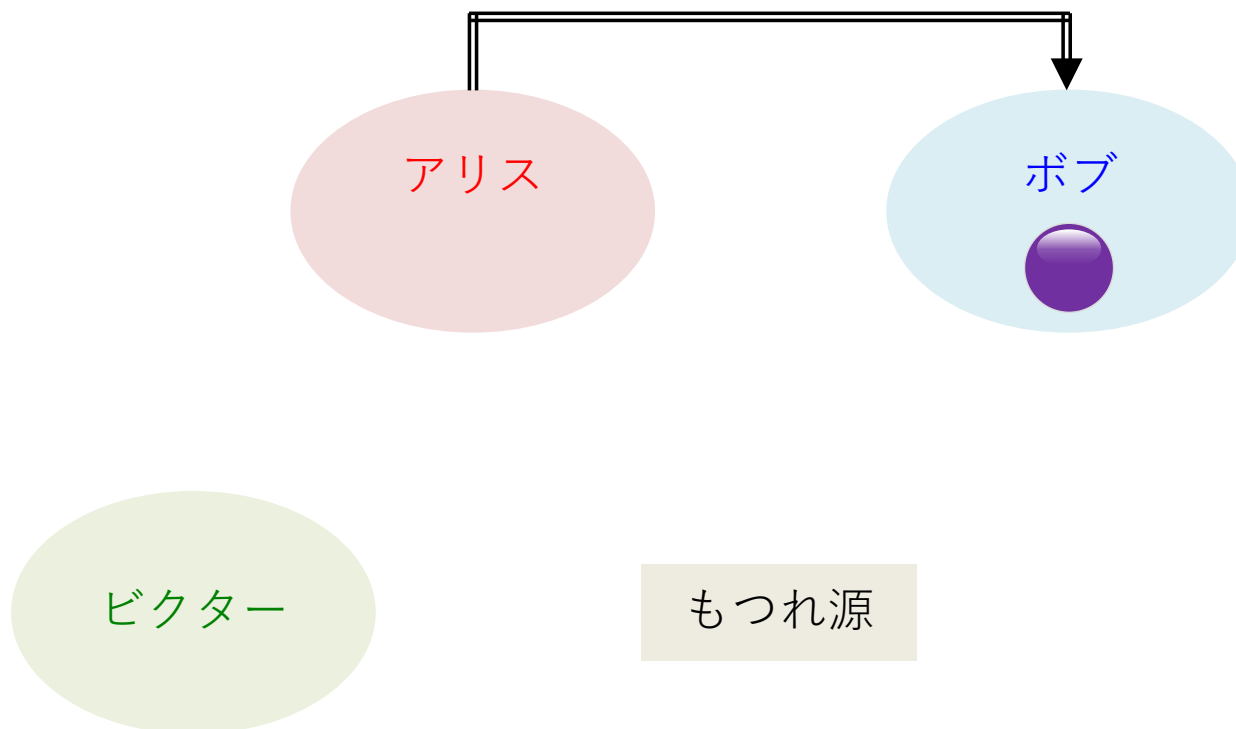
- ✓ アリスが結果 $xy = 01$ を得たとする
- ✓ ボブの状態は $X|\psi\rangle_B$ に確定するが、本人は知らない



ステップ3: 古典通信

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \cancel{\frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B} + \boxed{\frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B} + \cancel{\frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B} + \cancel{\frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B}$$

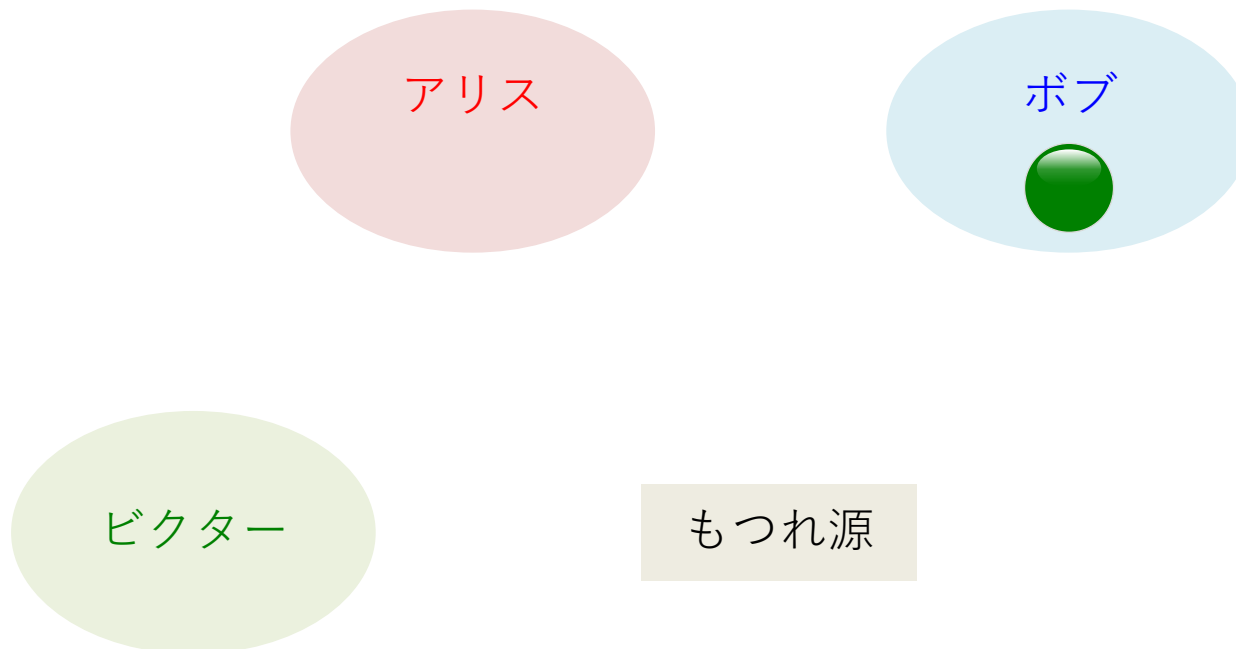
- ✓ アリスは古典通信により結果をボブに伝える



ステップ4: 復元

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B$$

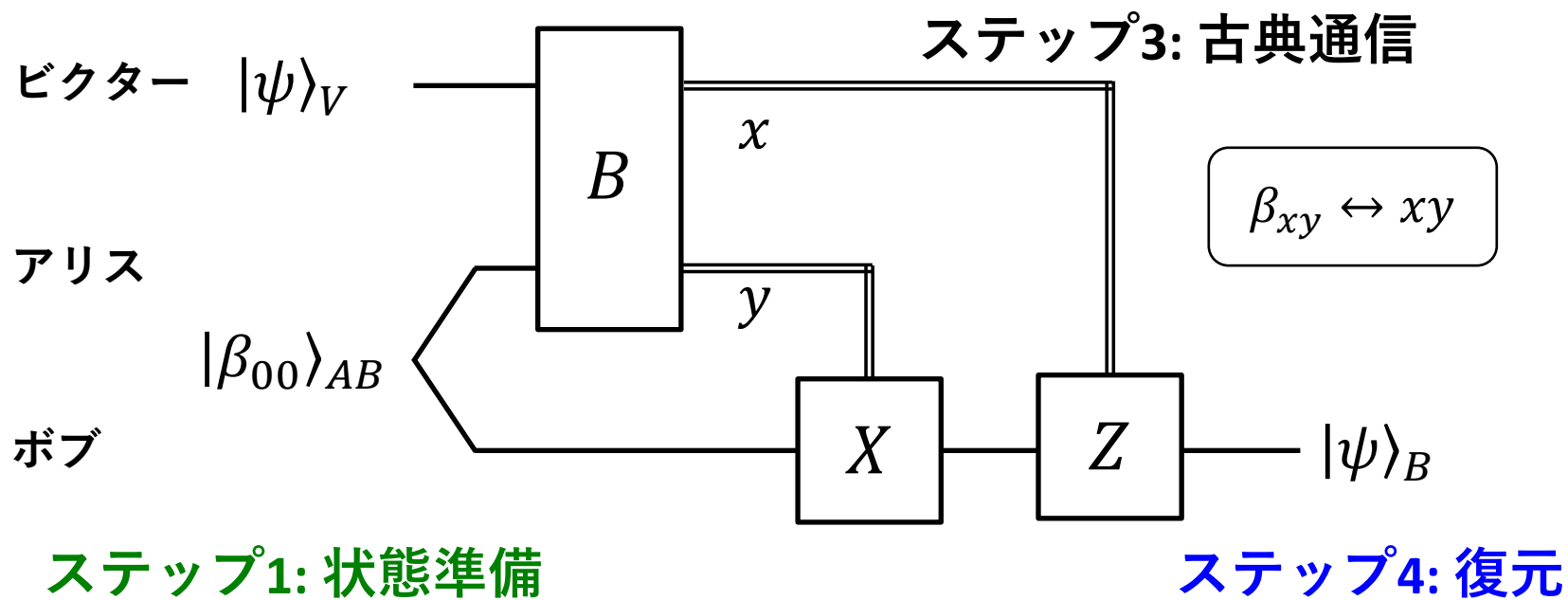
- ✓ ボブは必要な1量子ビットゲートを実行して状態を復元する



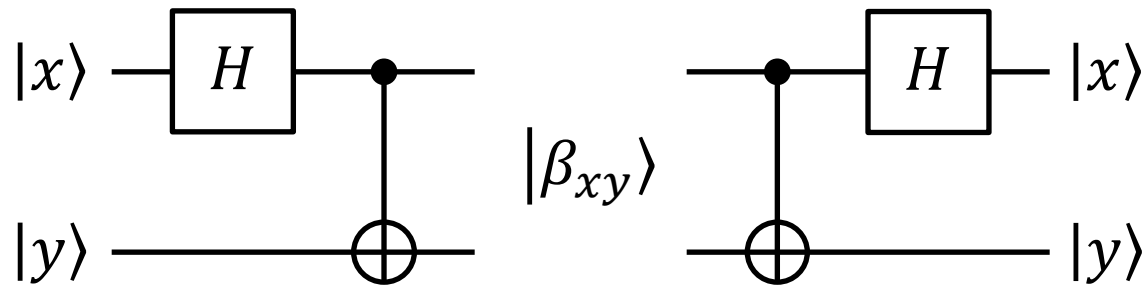
QTを量子回路で理解する

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B$$

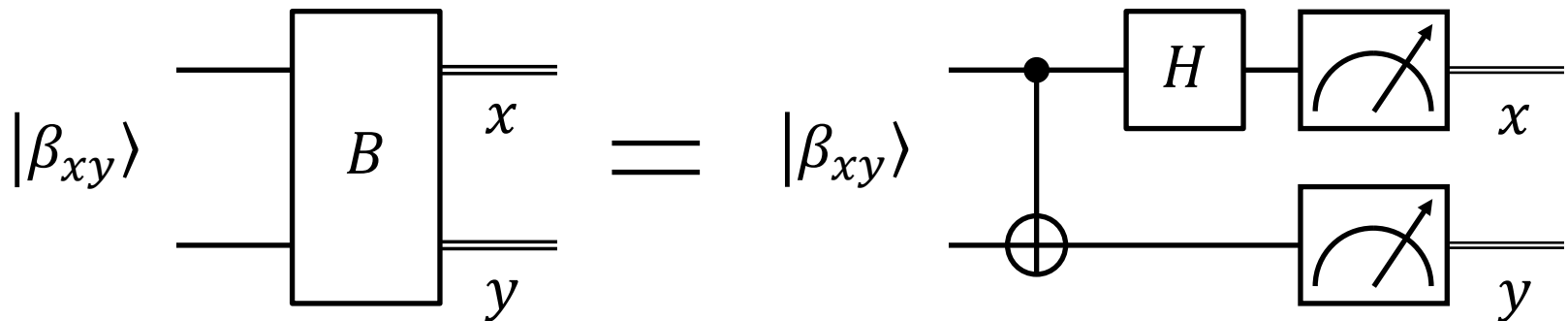
ステップ2: ベル測定



ベル状態生成と基底の変換



$$|xy\rangle \xleftarrow{H_1} \frac{|0y\rangle + (-1)^x |1y\rangle}{\sqrt{2}} \xleftarrow{C_{12}} \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} = |\beta_{xy}\rangle$$



測定と復元

$$|\psi\rangle_V |\beta_{00}\rangle_{AB} = \frac{1}{2} |\beta_{00}\rangle_{VA} |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{VA} X |\psi\rangle_B + \frac{1}{2} |\beta_{10}\rangle_{VA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{VA} XZ |\psi\rangle_B$$

$$= \frac{1}{2} \sum_{x,y} |\beta_{xy}\rangle_{VA} X^y Z^x |\psi\rangle_B$$

基底の変換

$$\frac{1}{2} \sum_{x,y} |xy\rangle_{VA} X^y Z^x |\psi\rangle_B$$

$$Z_B^x X_B^y$$

復元

$$\frac{1}{2} \sum_{x,y} |xy\rangle_{VA} |\psi\rangle_B$$

測定

$$x', y', X^{y'} Z^{x'} |\psi\rangle_B$$

$$Z_B^{x'} X_B^{y'}$$

復元

$$x', y', |\psi\rangle_B$$

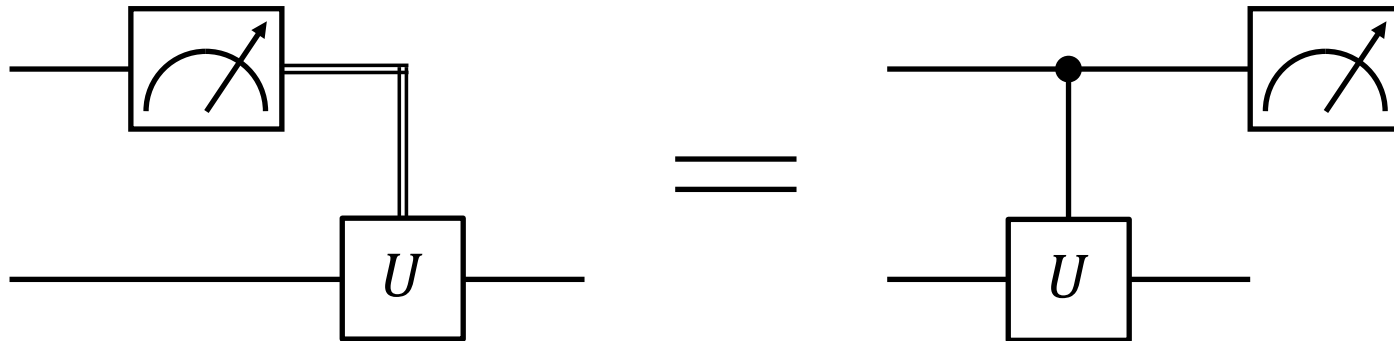
測定

測定と復元の順序は結果に影響しない

遅延測定の原理

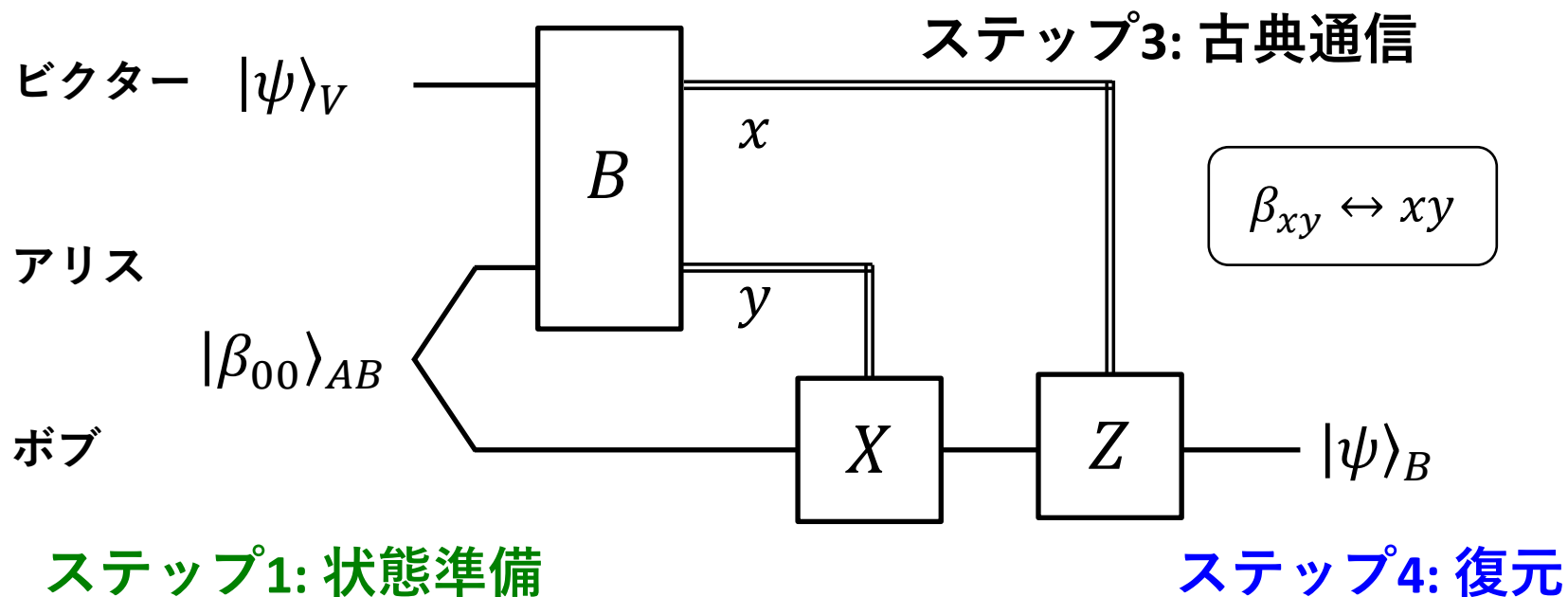
測定と制御ゲートは交換する

測定結果がそれ以降の回路で使われ続ける場合、
「測定 + 古典制御ゲート」を「量子制御ゲート」
に置き換え、最後に測定しても結果は変わらない



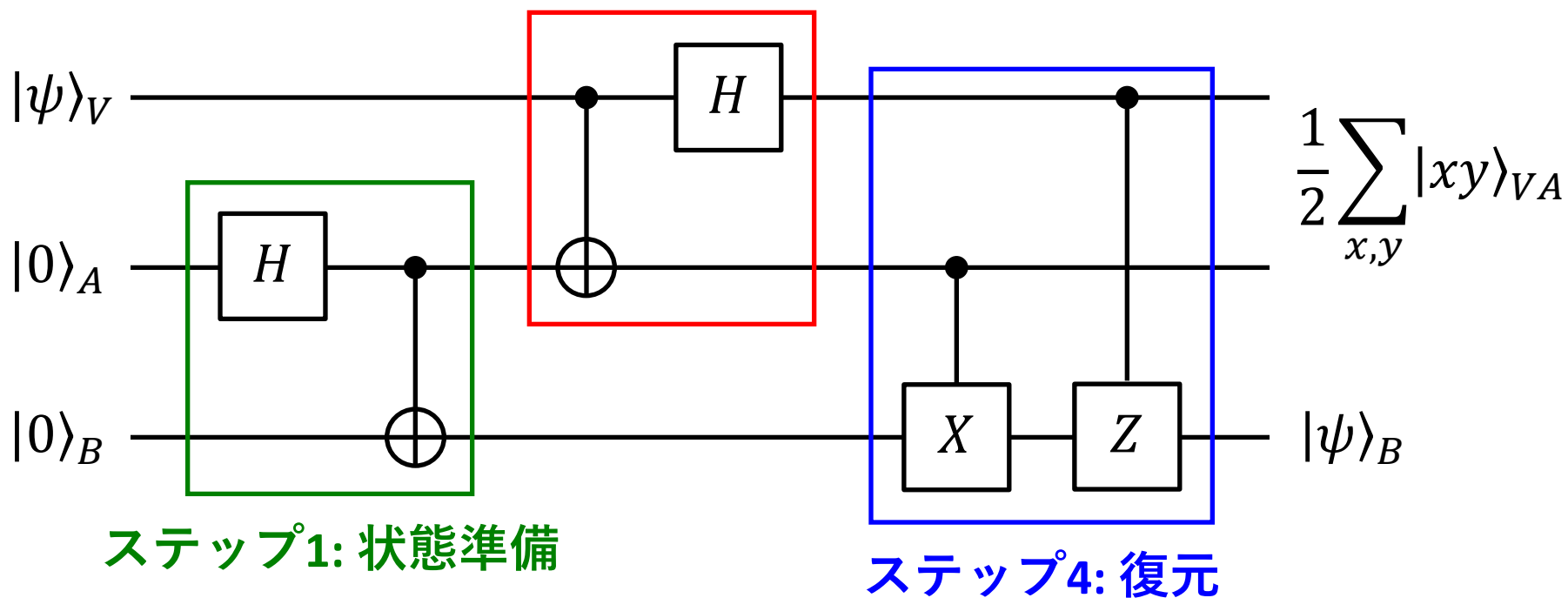
QTを量子回路で理解する

ステップ2: ベル測定



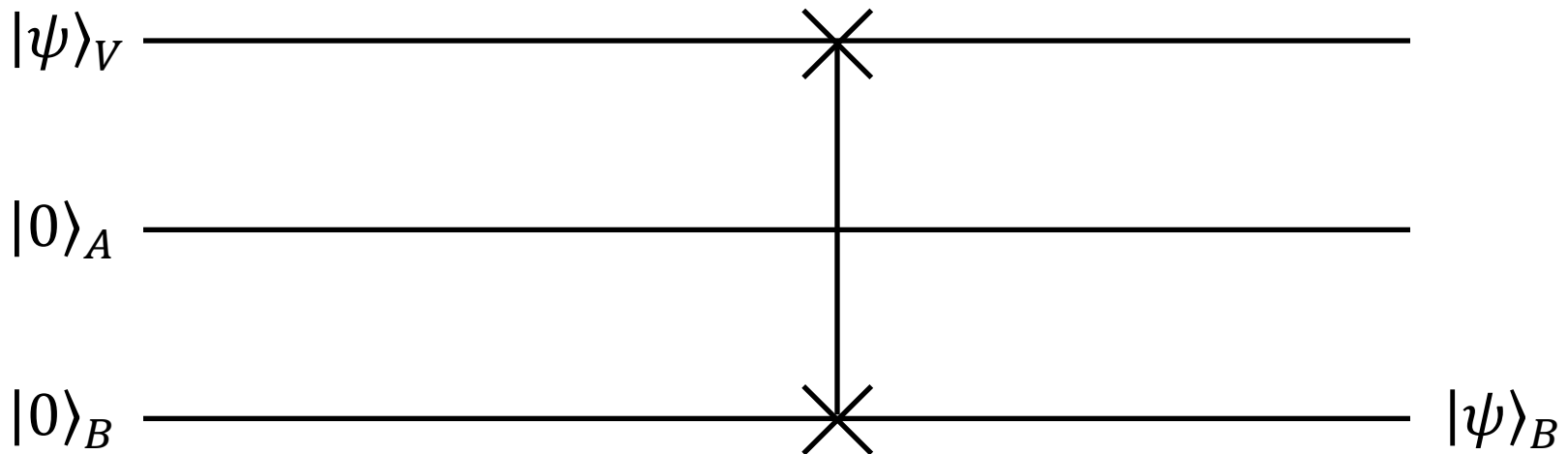
QTを量子回路で理解する

ステップ2: ベル測定



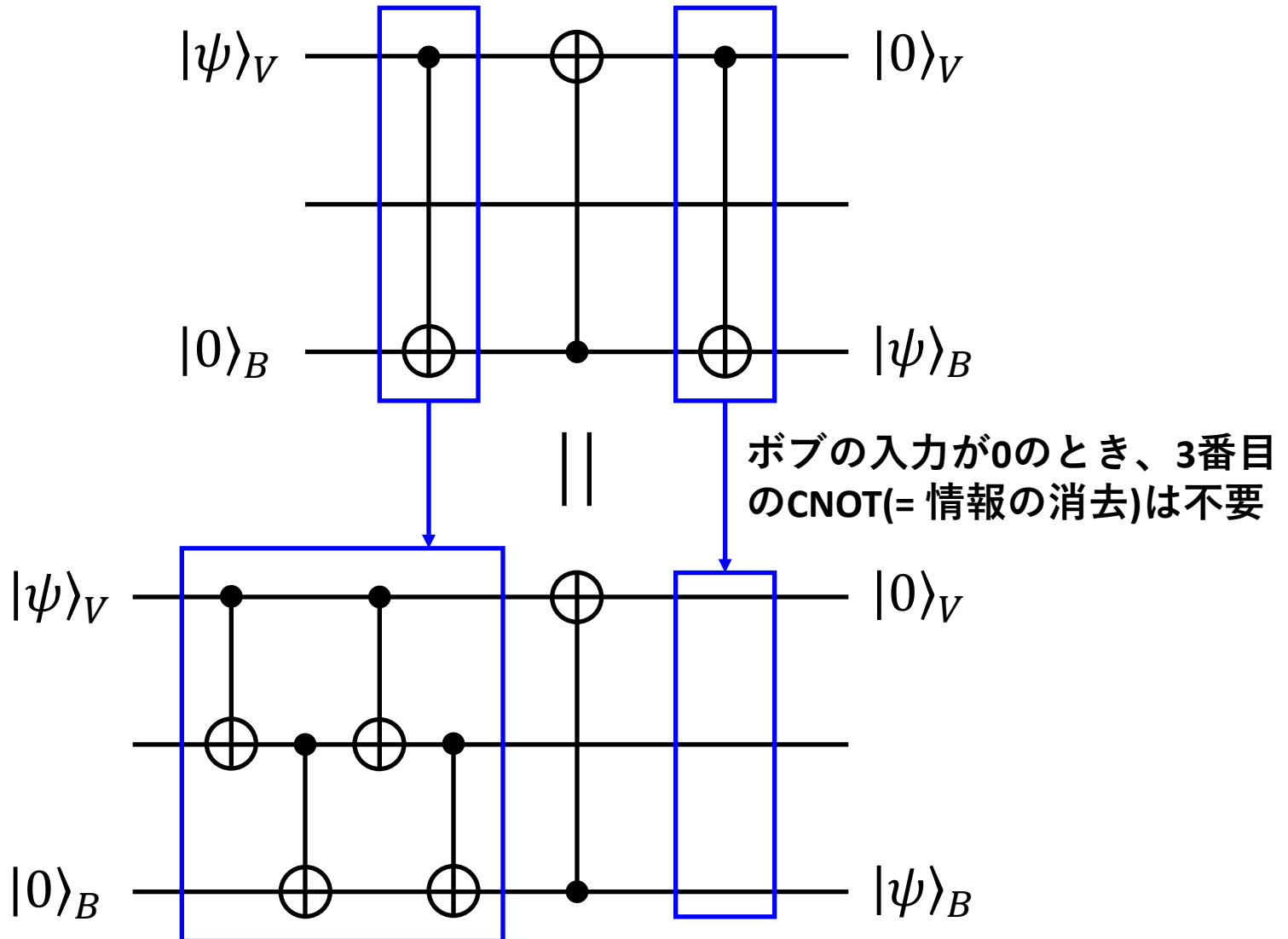
SWAP = QT?

- ✓ 古典通信なし
- ✓ アリスが関与しない

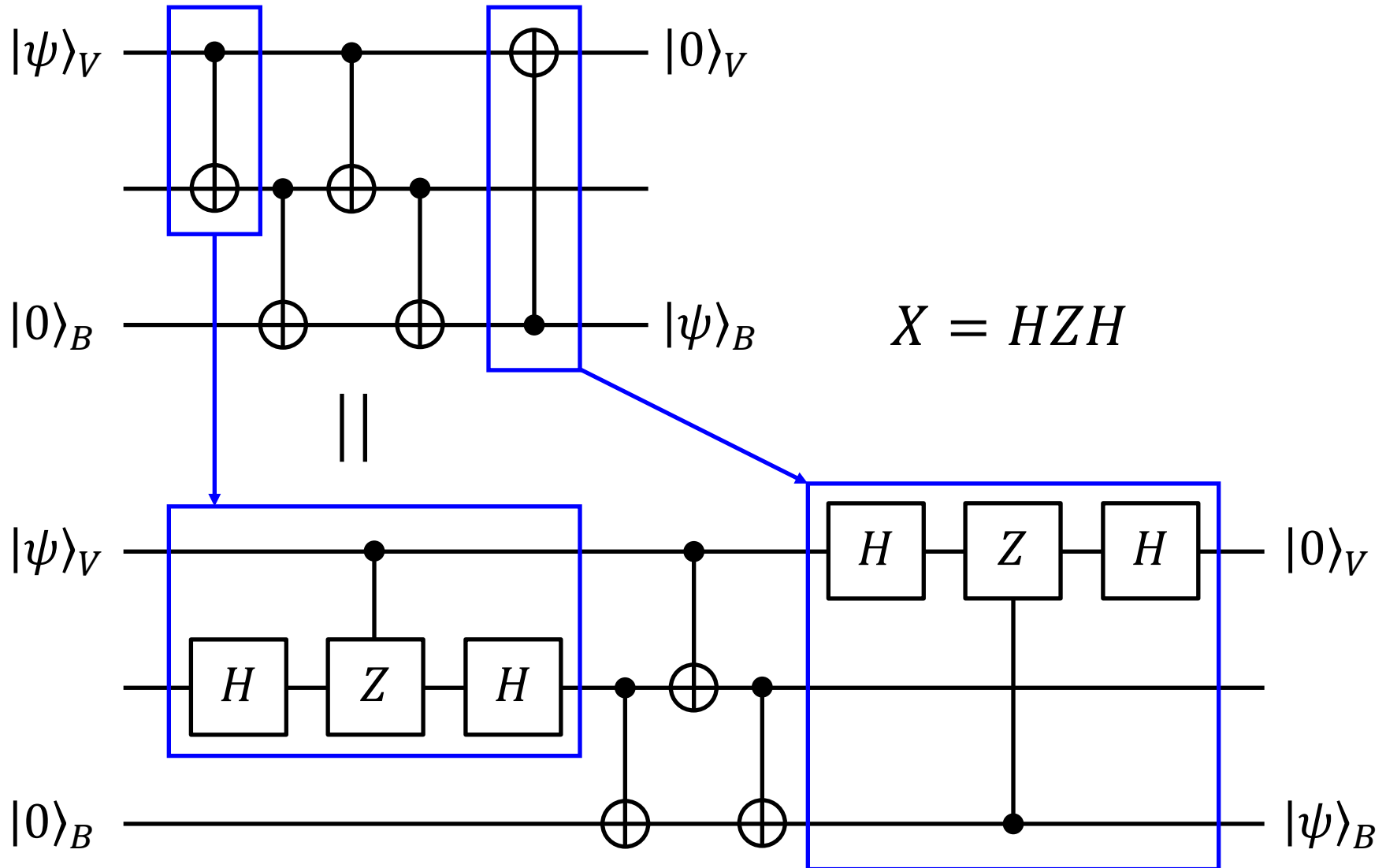


これからすること: SWAPを変形して、QTの量子回路にする

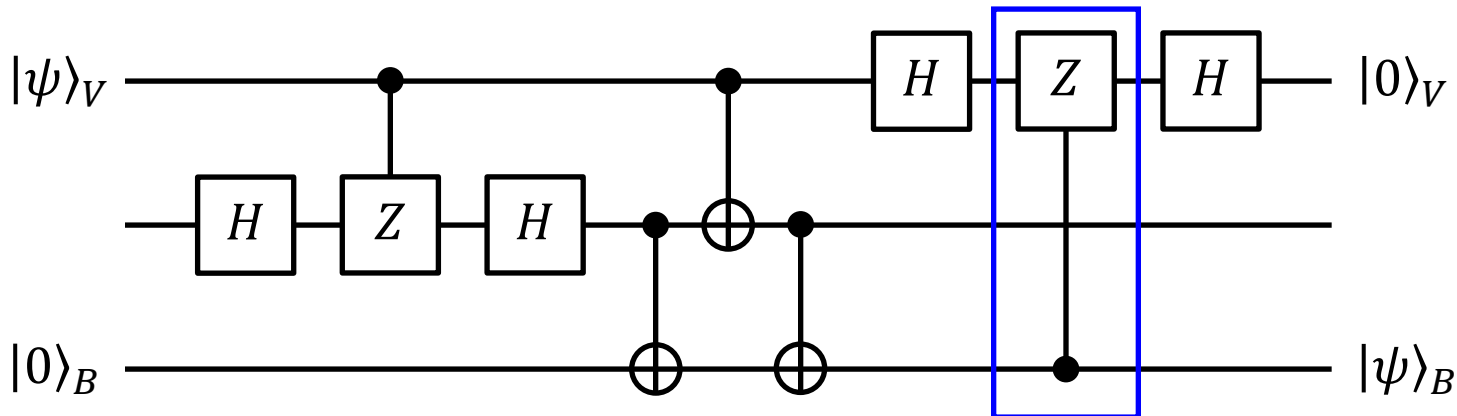
SWAPからQTへ



SWAPからQTへ

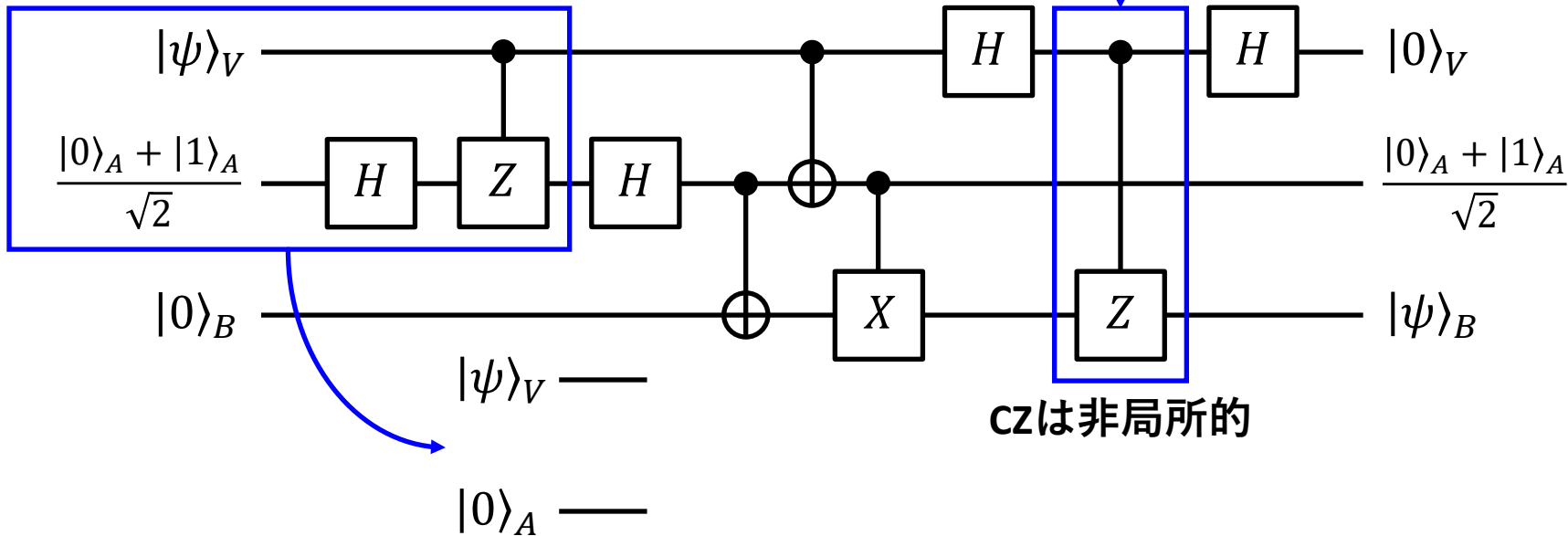


SWAPからQTへ

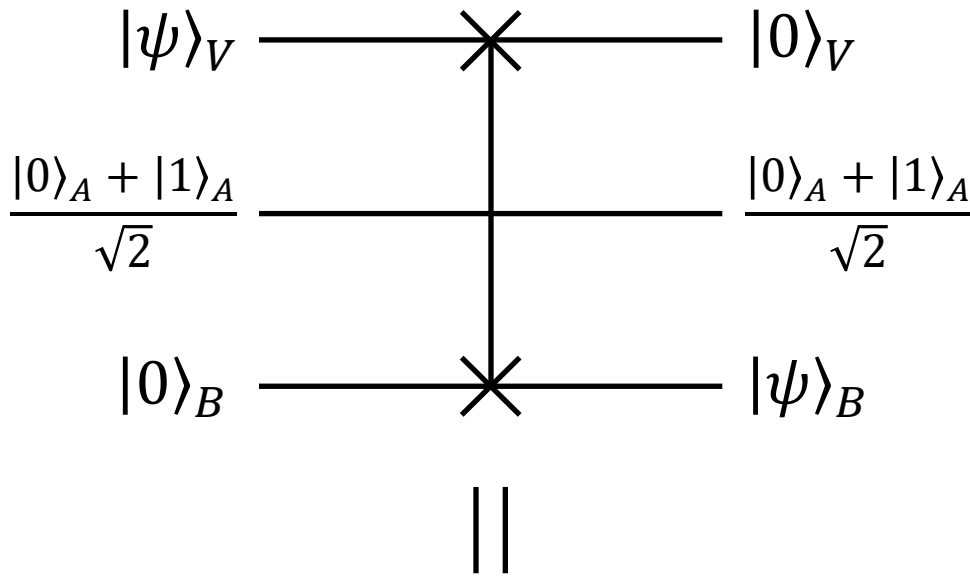


||

アリスの入力は任意



SWAPからQTへ



$$\frac{1}{2} (|00\rangle_{VA} + |01\rangle_{VA} + |10\rangle_{VA} + |11\rangle_{VA})$$

